

Geometric Complexity Theory V: Equivalence between blackbox derandomization of polynomial identity testing and derandomization of Noether's Normalization Lemma

Dedicated to Sri Ramakrishna

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Abstract

It is shown that the problem of derandomizing Noether's Normalization Lemma (NNL) in a strong form for the ring of invariants of the adjoint action of the general linear group on a tuple of matrices can be brought down from *EXPSPACE* (where it was earlier) to *PSPACE* unconditionally, to *PH* assuming the Generalized Riemann Hypothesis (GRH), and to *P* assuming the black-box derandomization hypothesis for symbolic trace identity testing (STIT) or equivalently symbolic determinant identity testing (SDIT). This derandomization problem lies at the heart of the wild problem of classifying tuples of matrices to which the problem of classifying representations of any (finite dimensional) algebra or quiver can be reduced. By the result here, black-box derandomization of STIT implies an explicit solution to a coarser version of this classification problem, namely, the problem of explicitly parametrizing and decomposing the semi-simple tuples (i.e. tuples with closed orbits), where the phrase explicit has a precise complexity theoretic meaning. Variants of the result are also shown assuming, instead of the black-box derandomization hypothesis, arithmetic lower bounds for constant depth circuits, or Boolean lower bounds for constant-depth threshold circuits or uniform Boolean conjectures in conjunction with GRH.

It is also shown that Noether's Normalization Lemma for the ring of invariants for any finite dimensional rational representation of the special linear group of fixed dimension can be quasi-derandomized in a strong form unconditionally, thereby bringing this problem from *EXPSPACE* (where it was earlier) to $\text{quasi-DET} \subseteq \text{quasi-NC}$.

A key ingredient of the proofs is to show that the varieties associated with the invariant rings under consideration are explicit, as per the general notion of explicit varieties introduced in this paper. It is shown that a strengthened form of the black-box derandomization hypothesis for polynomial identity testing (PIT) is equivalent to the problem of derandomizing Noether's Normalization Lemma in a strict form for general explicit varieties. It is also shown that a strengthened form of the black-box derandomization hypothesis for depth three arithmetic circuits is equivalent to the problem of derandomizing Noether's Normalization

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Lemma in a strict form for the higher order secant variety of the Chow variety. Finding explicit defining equations for this variety is a long-standing problem of invariant theory.

These and other results in this paper may explain in a unified way why proving lower bounds or derandomization results for arithmetic circuits of even depth three or constant-depth Boolean threshold circuits, or proving uniform Boolean conjectures without relativizable proofs has turned out to be so hard, and also why the classification problems of invariant theory and algebraic geometry have turned out to be so hard from the complexity-theoretic perspective. Thus this investigation reveals that the fundamental problems of Geometry (classification) and Complexity Theory (lower bounds and derandomization) share a common root difficulty that lies at the junction of these two fields. We call it the *GCT chasm*.

1 Introduction

Noether's Normalization Lemma (NNL), proved by Hilbert [Hil2], is the basis of a large number of foundational results in algebraic geometry such as Hilbert's Nullstellensatz. It also lies at the heart of the foundational classification problem of algebraic geometry. For any projective variety $X \subseteq P(K^k)$ of dimension n , where K is an algebraically closed field of characteristic zero and $P(K^k)$ is the projective space associated with K^k , the lemma says that any *random* (generic) linear map $\psi : K^k \rightarrow K^m$, for any $m \geq n + 1$, is regular (well defined) on X . Furthermore, for any such ψ , $\psi(X) \subseteq P(K^m)$, the image of X , is closed in $P(K^m)$, and (2) the fibre $\psi^{-1}(p)$, for any point $p \in \psi(X)$, is a finite set. In the context of the main results of this paper, k will be exponential in n and m will be polynomial in n . In this case Noether's Normalization Lemma expresses the variety X , embedded in the ambient space $P(K^k)$ of exponential dimension, as a finite cover of the variety $\psi(X)$, embedded in the ambient space $P(K^m)$ of polynomial dimension. This is its main significance from the complexity-theoretic perspective. By derandomization of Noether's Normalization Lemma we mean deterministic construction of ψ . We also refer to this problem as NNL in short. It turns out to be very difficult in general. First, the number of random bits used by the existing algorithms for general X is polynomial in k , the dimension of the ambient space K^k containing X . Since k is exponential in n , the number of random bits used is thus exponential in n . Second, for general X as above, the current best algorithms for even deterministic verification of ψ , let alone construction, based on a recent fundamental advance [MR2] in Gröbner basis theory, take in the worst case space that is polynomial in k and time that is exponential in k . The space bound for deterministic verification as well as construction is thus exponential in n and the time bound is double exponential in n . (Before [MR2] the time bound was double exponential in k , and hence, triple exponential in n .) Nothing better can be expected for general varieties because [MM, MR1] also prove a matching lower bound for the computation of Gröbner basis in this general setting.

We show that a stronger form of NNL that arises in the context of the problem of classifying tuples of matrices under the adjoint (simultaneous conjugate) action of the general linear group can be brought down from *EXPSPACE*, where it was earlier for the reasons above, to *PSPACE* unconditionally, to *PH* assuming the Generalized Riemann Hypothesis (GRH), and even further to *P* assuming the black-box derandomization hypothesis for symbolic trace identity testing (STIT) or equivalently symbolic determinant identity testing (SDIT). We specify the underlying variety using a *succinct specification* of bit-length linear in the dimension of the variety. The space and time bounds are in terms of the bit-length of the succinct specifi-

cation. The classification problem mentioned above is the well-known wild [Dz, DW] problem of invariant theory, also called an “impossible” or “hopeless” problem, to which the problem of classifying representations of any (finitely dimensional) algebra or quiver can be reduced. By the result here, black-box derandomization of STIT implies an explicit solution to a coarser version of this classification problem, namely, the problem of explicitly parametrizing and decomposing the semi-simple tuples (i.e. tuples with closed orbits), where the phrase explicit has a precise complexity theoretic meaning. Variants of this result are also shown assuming, instead of the black-box derandomization hypothesis, arithmetic lower bounds for constant depth circuits, or Boolean lower bounds for constant-depth threshold circuits or uniform Boolean conjectures in conjunction with GRH.

We also show that Noether’s Normalization Lemma for the ring of invariants for any finite dimensional rational representation of the special linear group of fixed dimension can be quasi-derandomized in a strong form (unconditionally), thereby bringing this problem from EX-PSPACE (where it was earlier for the same reasons) to quasi-DET \subseteq quasi-NC. This ring was the focus of Hilbert’s paper [H12] mentioned above. Noether’s Normalization Lemma was, in fact, proved therein to show that this ring is finitely generated.

A key ingredient in the proofs is to show that the varieties under consideration are *explicit*, as per the general notion of explicit varieties (Definition 10.2) formulated in this paper. We show that a strengthened form of the black-box derandomization hypothesis for polynomial identity testing (PIT) is equivalent to the problem of derandomizing Noether’s Normalization Lemma in a strict form for general explicit varieties. We also show that a strengthened form of the black-box derandomization hypothesis for depth three arithmetic circuits is equivalent to the problem of derandomizing Noether’s Normalization Lemma in a strict form for the higher order secant variety of the Chow variety. Finding explicit defining equations for this variety is a long-standing problem of invariant theory.

These and related results may explain in a unified way why proving lower bounds or derandomization results for arithmetic circuits in characteristic zero of even depth three or constant-depth Boolean threshold circuits, or proving uniform Boolean conjectures such as $EXP^{NP} \not\subseteq \Sigma_2^P \cap \Pi_2^P$ has turned out to be so hard, and also why the classification problems of invariant theory and algebraic geometry have turned out to be so hard from the complexity-theoretic perspective. Thus this investigation reveals that the fundamental problems of Geometry (classification) and Complexity Theory (lower bounds and derandomization) share a common root difficulty that lies at the junction of these two fields. We call it the *GCT chasm*.

On the negative side, the equivalence in this article says that black-box derandomization of PIT in characteristic zero would necessarily require proving, either directly or by implication, results in algebraic geometry that seem impossible to prove on the basis of the current knowledge. On the positive side, it says that if the fundamental derandomization and lower bound conjectures of complexity theory are true, then a large class of fundamental problems in algebraic geometry which appear intractable are actually tractable in the complexity-theoretic sense.

This article belongs to a series of articles on the GCT approach to the fundamental problems of complexity theory. See [Mu2] for an informal overview of the earlier articles in this series, and [Mu3] for a formal overview. Preliminary versions of the results in this article were announced

in [Mu4]. We now state the main results in this article in more detail.

1.1 The ring of matrix invariants

Let K be an algebraically closed field of characteristic zero. By black-box derandomization of PIT over K , we mean the problem of constructing an explicit hitting set [IW, HS, KI, Ag, RSh, SV, DSY] against all circuits over K and on r variables with size $\leq s$. By the size of a circuit, we mean the total number of edges in it. There is no restriction on the bit-lengths of the constants in the circuit. By an *explicit hitting set*, we mean a $\text{poly}(s)$ -time-constructible set $S_{r,s} \subseteq \mathbb{N}^r$ of test inputs such that, for every circuit C on K and r -variables $x = (x_1, \dots, x_r)$ with size $\leq s$, and $C(x)$ not identically zero, $S_{r,s}$ contains a test input b such that $C(b) \neq 0$. Here $C(x)$ denotes the polynomial computed by C . The fundamental black-box derandomization hypothesis in complexity theory [HS, IW, KI, Ag, RSh, SV, DSY] is that such explicit hitting sets exist. Black-box derandomization of PIT is important, because by the fundamental hardness vs. randomness principle [IW, HS, KI, Ag] it is essentially equivalent to proving circuit lower bounds for EXP, which are much easier variants of the nonuniform P vs. NP problem.

In our first result, we consider a restricted form of PIT called symbolic trace identity testing (STIT). It is equivalent [MP, Sp], up to polynomial factors, to the PIT for arithmetic branching programs or weakly skew straight-line programs, and also to the symbolic determinant identity testing (SDIT); cf. Section 2.1.

By a symbolic trace over K , we mean a polynomial of the form $\text{trace}(A(x)^l)$, where $A(x)$ is an $m \times m$ symbolic matrix whose each entry is a linear function over K in the variables $x = (x_1, \dots, x_r)$. By black-box derandomization of STIT over K , we mean the problem of constructing an explicit ($\text{poly}(m, r)$ -time computable) hitting set $S_{r,m}$ of test inputs in \mathbb{N}^r such that, for any symbolic trace polynomial $\text{trace}(A(x)^l)$, $l \leq m$, that is not identically zero, there exists a test input $b \in S_{r,m}$ such that $\text{trace}(A(b)^l) \neq 0$.

Let $M_m(K)$ be the space of $m \times m$ matrices over K , and $V = M_m(K)^r$, the direct sum of r copies of $M_m(K)$, with the adjoint (simultaneous conjugate) action of $G = SL_m(K)$:

$$(A_1, \dots, A_r) \rightarrow (PA_1P^{-1}, \dots, PA_rP^{-1}), \quad (1)$$

where $A_1, \dots, A_r \in M_m(K)$ and $P \in SL_m(K)$. Let U_1, \dots, U_r be variable $m \times m$ matrices. Then the coordinate ring $K[V]$ of V can be identified with the ring $K[U_1, \dots, U_r]$ generated by the variable entries of U_i 's. Let $K[V]^G \subseteq K[V]$ be the ring of invariants with respect to the adjoint action. By an invariant we mean a polynomial $f(U_1, \dots, U_r)$ in the variable entries of U_i 's such that

$$f(U_1, \dots, U_r) = f(PU_1P^{-1}, \dots, PU_rP^{-1}),$$

for all $P \in SL_m(K)$. Let $n = \dim(V) = rm^2$. It is known that $K[V]^G$ is finitely generated [H12, Pr, Rz]. Hence, it can be thought of as the coordinate ring of the variety $V/G = \text{spec}(K[V]^G)$, called the categorical quotient [MFK]. We specify V and G *succinctly* by the pair (m, r) in unary. This pair will be the input in our problem. Thus the bit-length of this succinct input specification is $O(n)$. All the space and time bounds for the algorithms with this input will be in terms of n .

By Noether's Normalization Lemma (Lemma 2.10), there exists a set $S \subseteq K[V]^G$ of $\text{poly}(n)$ homogeneous invariants such that $K[V]^G$ is integral over the subring generated by S . (This statement of Noether's Normalization Lemma is equivalent to the one given in the beginning of the introduction. Here a ring R is said to be integral over its subring T if every $r \in R$ satisfies a monic polynomial equation of the form $r^l + b_{l-1}r^{l-1} + \dots + b_1r + b_0 = 0$, where each $b_i \in T$.) In fact, there even exists such an S of optimal cardinality equal to $\dim(K[V]^G) \leq n$. It is known that any suitably randomly chosen S of this cardinality has the required property. Such an S of optimal cardinality is called an h.s.o.p. (homogeneous system of parameters) of $K[V]^G$. By the problem of derandomizing Noether's Normalization Lemma for $K[V]^G$, in short NNL, we mean the problem of constructing an S of $\text{poly}(n)$ cardinality such that $K[V]^G$ is integral over the subring generated by S .

Following [DK] (cf. Section 2.3.2 therein), call a set $S \subseteq K[V]^G$ separating if for any $u, v \in V$ such that $w(u) \neq w(v)$, for some $w \in K[V]^G$, there exists an $s \in S$ such that $s(u) \neq s(v)$. Noether's Normalization Lemma for certain varieties $X_j[V, G]$, $1 \leq j \leq m^2$, associated with V and G implies that a small separating S of $\text{poly}(n)$ cardinality exists; cf. Proposition 2.12. By the problem of derandomizing Noether's Normalization Lemma for $K[V]^G$ in a strong form we mean the problem of constructing a separating S of $\text{poly}(n)$ cardinality. If S is separating, then $K[V]^G$ is automatically integral over the subring generated by S ; cf. Theorem 2.3.12 in [DK]. Thus the construction of a small separating S is a stronger form of derandomization of Noether's Normalization Lemma for V/G .

This stronger form of NNL lies at the heart of the problem of classifying r -tuples of $m \times m$ matrices over K under simultaneous conjugation by $SL_m(K)$. When $r = 1$, this classification is provided by the Jordan normal form. For $r \geq 2$, this is the so-called wild classification problem of invariant theory. The phrase "wild" here (meaning "impossible" or "hopeless") has a precise mathematical meaning [Dz, BSe, DW] analogous to the phrase "NP-complete" in complexity theory. There is also a result [BSe] analogous to the NP-completeness result which says that a solution to the problem of classifying tuples of matrices implies a solution to the problem of classifying representations of any quiver. By [MFK], the points of $V/G = \text{spec}(K[V]^G)$ are in one-to-one correspondence with the closed G -orbits in V . The points in V with closed orbits are called stable [MFK] or semi-simple. If we restrict ourselves to only closed orbits as in [MFK], then a coarser classification problem is to (1) explicitly parametrize the closed orbits, i.e., associate with each closed orbit an explicit point in a one-to-one (regular) manner so that the set of such explicit points is a closed variety embedded in an ambient space of small ($\text{poly}(n)$) dimension, and (2) explicitly decompose any semi-simple tuple into simple tuples. By [Km], each point $v \in V$ can be driven to the unique closed orbit in the G -orbit closure of v by an optimal one parameter subgroup. One also wants to associate with v the instability flag of this optimal one parameter subgroup explicitly. In the mathematics literature, the phrases "explicit" and "impossible" are used informally. We give them formal interpretation from the complexity-theoretic perspective as follows.

By explicit, we mean $\text{poly}(n)$ -time computable. Specifically, by [MFK], invariants distinguish closed G -orbits in V . Hence a separating S distinguishes closed orbits. A small separating S yields a one-to-one and onto regular map from the closed orbits of V to the points of a closed variety embedded in an ambient space of $\text{poly}(n)$ dimension; cf. the proof of Theorem 3.5. By an *explicit* parametrization of closed orbits we mean parametrization by a small separating S that

can be computed in $\text{poly}(n)$ time. The problem of computing such an explicit small separating S is precisely the stronger form of NNL.

The existing techniques for Noether normalization can be applied to V/G to construct an h.s.o.p. for $K[V]^G$. Since the dimension of the ambient space containing V/G is exponential in n , these techniques take space that is exponential in n and time that is double exponential; cf. Proposition 3.1. The existing techniques for Noether normalization can also be applied to the varieties $X_j[V, G]$'s to construct a separating homogeneous S of small degree and $\text{poly}(n)$ cardinality (and even optimal cardinality) deterministically; cf. Proposition 3.2. These techniques also take space that is exponential in n and time that is double exponential in n since the dimension of the ambient space containing $X[V_j, G]$ is also exponential in n . We interpret the phrase “impossible” as referring to the seemingly impossible task of bringing this double exponential time bound down to polynomial.

This may well be impossible if we insist on optimality, as done quite often in the mathematics literature, and ask for an h.s.o.p. or a separating S of optimal cardinality, since it is not even known if such optimal objects with subexponential bit-length of specification exist. Furthermore, it seems difficult to improve on the current exponential space and double exponential time bound with the existing techniques if we want a separating homogeneous S of small degree and optimal cardinality. But if we do not insist on optimality and only require S to be small of $\text{poly}(n)$ cardinality then the following result (Theorem 1.1) says that this double exponential time bound can be brought down to polynomial assuming black-box derandomization of STIT. Specifically, it says that the problem of constructing such an S can then be brought down from EXPSPACE, where it was earlier for the reasons above, to PSPACE unconditionally, to PH assuming GRH, and even further to P assuming the black-box derandomization hypothesis for STIT. The space and time bounds are in terms of the bit-length n of the succinct specification.

We need a few definitions for stating this result. Call a set $S \subseteq K[V]^G$ an *s.s.o.p.* (*small system of parameters*) for $K[V]^G$ if (1) S contains $\text{poly}(n)$ homogeneous invariants of $\text{poly}(n)$ degree, (2) $K[V]^G$ is integral over the subring generated by S , (3) each invariant $s = s(U_1, \dots, U_r)$ in S has a weakly skew straight-line program [MP] over \mathbb{Q} and the variable entries of U_i 's of $\text{poly}(n)$ bit-length. Here (3) is equivalent [MP] to (3)': every $s \in S$ can be expressed as the determinant of a matrix of $\text{poly}(n)$ size whose entries are (possibly non-homogeneous) linear combinations of the entries of U_i 's with rational coefficients of $\text{poly}(n)$ bit length. By [Cs, MP], it follows that, given such a weakly-skew straight-line program of an invariant $s \in S$ and any rational matrices $A_1, \dots, A_r \in M_m(\mathbb{Q})$, the value $s(A_1, \dots, A_r)$ can be computed in time polynomial in n and the total bit-length of the specifications of A_i 's (and even fast in parallel). Thus an s.s.o.p. is an approximation to h.s.o.p. that has a small specification and is easy to evaluate. Henceforth, we only require that S be small (of $\text{poly}(n)$ cardinality) and do not insist on optimality. We say that S is a separating s.s.o.p. if (2) is replaced by the stronger (2)': S is separating.

Let us call a set S an *e.s.o.p.* (*explicit system of parameters*) for $K[V]^G$ if (1) S is an s.s.o.p. for $K[V]^G$, and (2) given m and r , the specification of S , consisting of a weakly skew straight-line program as above for each $s \in S$, can be computed in $\text{poly}(n)$ time. Thus the key difference between an s.s.o.p. and an e.s.o.p. is that an e.s.o.p. can also be constructed *uniformly*. We call S a separating e.s.o.p. if (1) is replaced by the stronger (1)': S is a separating s.s.o.p. A separating e.s.o.p. yields an explicit parametrization of the closed G -orbits in V ; cf. Theorem 1.1 (b) (1) be-

low. A (separating) quasi-e.s.o.p. is defined by replacing the $\text{poly}(n)$ bounds by $O(2^{\text{poly}(\log(n))})$ bounds. A (separating) subexponential-e.s.o.p. with a small exponent $\delta > 0$ is defined by replacing the $\text{poly}(n)$ bounds by $O(2^{O(n^\delta)})$ bounds. Quasi-s.s.o.p. and subexponential-s.s.o.p. (separating) are defined similarly.

By the following result, black-box derandomization of STIT implies existence of a separating e.s.o.p. and thus explicit coarse classification (parametrization and decomposition) of semi-simple tuples and GRH implies a *PH*-algorithm for the construction of a separating s.s.o.p. See [AB] for the precise definitions of the various complexity classes used in this result.

Theorem 1.1 *Let K be an algebraically closed field of characteristic zero. Let V and G be as above.*

(a) *A separating s.s.o.p. for $K[V]^G$ exists and can be constructed in $\text{poly}(n)$ work space unconditionally¹.*

(b) (1) *A separating e.s.o.p. yields a one-to-one and onto explicit (polynomial time computable) regular map from the closed orbits of V to the points of a closed variety embedded in an ambient space of $\text{poly}(n)$ dimension.* (2) *Assuming the black-box derandomization hypothesis for STIT over K , $K[V]^G$ has a separating e.s.o.p. Analogous result holds, after replacing the $\text{poly}(n)$ bound everywhere by $O(n^{O(\log n)})$ bound, if we assume black-box derandomization over K of PIT for depth four circuits instead of STIT.*

(c) *Suppose STIT over K for $m \times m$ matrices and r variables has $O(2^{n^\epsilon})$ -time-computable hitting set, $n = rm^2$, for any small constant $\epsilon > 0$. Then $K[V]^G$ has a subexponential separating e.s.o.p. for any exponent $\delta > 0$.*

(d) *Suppose GRH holds. Then: (1) The problem of constructing a separating s.s.o.p. for $K[V]^G$ belongs to $\Sigma_3 \subseteq PH$. (2) It also belongs to $\Delta_3^{\text{quasi}P} \subseteq \text{quasi-PH}$. This means there is a quasi-polynomial time algorithm, with an access to the NP^{NP} -oracle, for constructing a separating quasi-s.s.o.p. for $K[V]^G$. (3) It also belongs to $i.o.MA_{\text{quasi-half-EXP}}$. This means there is an $MA_{\text{quasi-half-EXP}}$ -algorithm for constructing a separating quasi-half-exponential s.s.o.p. (as defined in Section 7) that works correctly for infinitely many n .*

(e) *Given a rational $v \in V$, the instability flag $[Km] \phi(v)$ of an optimal one-parameter subgroup that drives v to a point in the unique closed orbit that lies in the closure of the G -orbit of v can be computed in time that is polynomial in n and the bit-length of the specification of v .*

(f) *Given a semi-simple rational $v \in V$, the decomposition of v into simple tuples can be computed in time that is polynomial in n and the bit-length of the specification of v .*

Here, and in what follows, the prefix i.o. denotes the infinitely-often version [KI] of the complexity class. The complexity class $MA_{\text{quasi-half-EXP}}$ in (d) denotes the quasi-half-exponential [MVW] variant of MA, and we think of MA as a class of functions rather than decision problems. It is interesting to know how far GRH can take us here in comparison to the black-box

¹We realized that the key ingredient in the proof of existence of a (separating) s.s.o.p. as in (a), namely that $V/G = \text{spec}(K[V]^G)$ is an explicit variety (Lemma 3.7), can be used to construct a (separating) s.s.o.p. in polynomial space unconditionally after the publication of the preliminary abstract [Mu4]. As such, the unconditional known space complexity of the problem of constructing an s.s.o.p. for this ring (as well the coordinate rings of general explicit varieties) was stated in that abstract as exponential, as it was then to our knowledge.

derandomization hypothesis for STIT, since this may give us some idea of the difficulty of the black-box derandomization hypothesis. Thus we may ask if NNL for $K[V]^G$ can be put in P or even NP (thought as a class of functions) assuming GRH . That is open. Theorem 1.1 (d) (2) and (3), the main technical results in this paper, are attempts to go as far as we can on the basis of GRH with this in mind. It may be possible to strengthen the proof of (d) (3) to show that NNL for $K[V]^G$ is in MA assuming GRH ; cf. the remark after Theorem 7.5. But as far as we can see on the basis of the proof of (d) (3), GRH may not suffice to put the problem in NP .

The variety $V/G = \text{spec}(K[V]^G)$ here is a basic prototype of the moduli space of representations of a *wild quiver* [DW]. Theorems 1.1 (a)-(d) can be generalized to moduli spaces of representations of any wild quiver. By this generalization (Theorem 4.1), black-box derandomization of STIT over an algebraically closed field K of characteristic zero implies derandomization of Noether's Normalization Lemma in a strong form, and thereby, explicit parametrization of semi-simple representations for any wild quiver and dimension data. Analogous result holds for semi-simple representations of any finite dimensional algebra; cf. Theorem 3.14.

The following result is a variant of Theorem 1.1 (b) and (c) assuming, instead of the black-box derandomization hypothesis, arithmetic lower bounds, or Boolean lower bounds in conjunction with GRH .

Let nonuniform $NC[d(n), S(n)]$ denote the class of Boolean functions that can be computed by nonuniform Boolean circuits of depth $d(n)$ and size $S(n)$, and $TC^0[S(n)]$ the class of Boolean functions that can be computed by constant-depth Boolean threshold circuits of size $S(n)$. Let nonuniform NC^i denote nonuniform $NC[\log^i n, \text{poly}(n)]$.

Theorem 1.2 *Let V, G , and K be as in Theorem 1.1.*

(a) *Suppose EXP has a multilinear integral function with a subexponential arithmetic lower bound. Then $K[V]^G$ has a separating quasi-e.s.o.p.*

The arithmetic lower bound assumption here is that there exists an exponential-time-computable multilinear polynomial $f(x_1, \dots, x_r)$ with integral coefficients of $\text{poly}(r)$ bit-length such that f cannot be computed by an arithmetic circuit over K of (1) $O(2^{r^a})$ size and $O(r^a)$ depth, for some constant $a > 0$, or alternatively, (2) $O(2^{r^{a'}})$ size and constant depth for some constant $a' > 0$.

(b) *Suppose EXP has a multilinear integral function with a polynomial arithmetic lower bound. Then $K[V]^G$ has a separating subexponential-e.s.o.p for any exponent $\delta > 0$.*

The arithmetic lower bound assumption here is that f as in (a) is not in algebraic- NC^2 ; i.e., it cannot be computed by an arithmetic circuit over K of $O(\log^2 r)$ depth and $O(r^a)$ size for any constant $a > 0$.

(c) *Suppose EXP has a subexponential Boolean lower bound and that GRH holds. Then $K[V]^G$ has a separating quasi-e.s.o.p.*

*The Boolean lower bound assumption here is that $EXP \not\subseteq \text{i.o. nonuniform-NC}[n^\epsilon, 2^{n^\epsilon}]$, for some constant $\epsilon > 0$, or alternatively, that $EXP \not\subseteq \text{i.o. TC}^0[2^{n^{a'}}]$, for some constant $a' > 0$. The prefix *i.o.* can be removed from the assumptions and added to the conclusions.*

(d) *Suppose EXP has a polynomial Boolean lower bound and that GRH holds. Then $K[V]^G$ has a separating subexponential-e.s.o.p. for any exponent $\delta > 0$.*

The Boolean lower bound assumption here is that (1) $EXP \not\subseteq i.o. \text{ nonuniform-NC}^3$, or alternatively, that (2) $EXP \not\subseteq i.o. TC^0[n^{O(\sqrt{n} \log n)}]$. The prefix *i.o.* can be removed from the assumptions and added to the conclusions.

(e) Suppose $EXP \not\subseteq MA$ and that GRH holds. Then there is a subexponential time algorithm for constructing a separating subexponential-s.s.o.p. for $K[V]^G$, for any exponent $\delta > 0$, that is correct for infinitely many n .

(f) The results in (a)-(e) also hold, replacing EXP by EXP^{NP} in the assumptions and giving the algorithm for constructing S an access to the NP-oracle. The uniform lower bound assumption in (e) is replaced in this case by the uniform assumption that $EXP^{NP} \not\subseteq \Sigma_2^P \cap \Pi_2^P$, or alternatively, that $NEXP \not\subseteq MA$.

If we fix a conjecturally hard function in EXP , such as the permanent, then the proof of Theorem 1.2 (a), in conjunction with the arithmetic NW-generator [NW, KI], yields a concrete conjecturally correct separating quasi-e.s.o.p., and thereby, a conjecturally correct quasi-explicit parametrization of semi-simple tuples of matrices, and more generally, semi-simple representations of any finite dimensional algebra; cf. Theorem 5.3.

1.2 The general ring of invariants

Next we turn to the general setting when V is any rational representation of $G = SL_m(K)$ of dimension n .

Since G is reductive [Fu], V can be decomposed as a direct sum of irreducible representations of G :

$$V = \sum_{\lambda} m(\lambda) V_{\lambda}(G). \quad (2)$$

Here $\lambda : \lambda_1 \geq \dots \geq \lambda_l, l < m$, is a partition, i.e., a sequence of non-negative integers, and $V_{\lambda}(G)$ is the irreducible representation of G (Weyl module [Fu]) labelled by λ . We specify V and G *succinctly* by giving n and m (in unary) and the multiplicities $m(\lambda)$'s (in unary) for all λ 's that occur with nonzero multiplicity in this decomposition. The *degree* d of V is defined to be $\max\{|\lambda| = \sum \lambda_i\}$, where λ ranges over such partitions. Fix the standard monomial basis [DRS, LR] for each $V_{\lambda}(G)$ and thus a standard monomial basis for V . Let v_1, \dots, v_n be the coordinates of V for this basis. This fixes the action of G on V . Let $K[V] = K[v_1, \dots, v_n]$ denote the coordinate ring of V . Let $K[V]^G$ be its subring of G -invariants. We call a polynomial $f(v) \in K[V]$ a G -invariant if $f(\sigma^{-1}v) = f(v)$ for all $\sigma \in G$. By NNL in this context, we mean the problem of deterministically constructing an S of $\text{poly}(n)$ cardinality such that $K[V]^G$ is integral over the subring generated by S . The bit-length of the succinct specification of $V/G = \text{spec}(K[V]^G)$, consisting of n and m in unary and $m(\lambda)$'s in unary, is $O(n + m)$. The existing techniques for Noether normalization applied to V/G take space that is exponential in the bit-length $O(n + m)$ of the succinct specification and time that is double exponential even if m is constant; cf. Section 2.4 and the remarks after Theorem 1.4. The following result (Theorem 1.3) says that this double exponential time bound can be brought down to quasi-polynomial unconditionally for constant m .

We say that $S \subseteq K[V]^G$ is an *s.s.o.p.* (*small system of parameters*) for $K[V]^G$ if (1) $K[V]^G$ is integral over the subring generated by S , (2) the cardinality of S is $\text{poly}(n, m)$, (3) every

invariant in S is homogeneous of $\text{poly}(n, m)$ degree, and (4) every $s \in S$ has a small specification in the form of a weakly skew straight-line program [MP] of $\text{poly}(n, m)$ bit-length over \mathbb{Q} and the coordinates v_1, \dots, v_n of V . Here (4) is equivalent [MP] to (4)': every $s \in S$ can be expressed as the determinant of a matrix of $\text{poly}(n, m)$ size whose entries are (possibly non-homogeneous) linear combinations of v_1, \dots, v_n with rational coefficients of $\text{poly}(n, m)$ bit length. A separating s.s.o.p. is defined by replacing (1) with the stronger (1)': S is separating [DK]. We define a (separating) s.s.o.p. in a relaxed sense by dropping the weakly skew requirement in (4) and the degree requirement in (3).

We call a subset $S \subseteq K[V]^G$ an *e.s.o.p.* (*explicit system of parameters*) for $K[V]^G$ if (1) S is an s.s.o.p. for $K[V]^G$, and (2) the specification of S , consisting of a weakly skew straight-line program as above for each $s \in S$, can be computed in $\text{poly}(n, m)$ time, given n, m , and the nonzero multiplicities $m(\lambda)$'s of $V_\lambda(G)$'s as in eq.(2). A separating e.s.o.p. is defined by replacing (1) with the stronger: (1) S is a separating s.s.o.p. for $K[V]^G$. A (separating) e.s.o.p. in a relaxed sense is defined by requiring S to be a (separating) s.s.o.p. in a relaxed sense only. A (separating) quasi-e.s.o.p. (in a relaxed sense) is defined similarly.

If G has constant dimension, then Noether's Normalization Lemma for $K[V]^G$ can be quasi-derandomized in a strong form unconditionally, and parametrization of closed orbits can also be done explicitly:

Theorem 1.3 *Suppose K is an algebraically closed field of characteristic zero. Let V as in (2) be a rational representation of $G = SL_m(K)$ of dimension n . Suppose m is constant, or more generally, $O(\text{polylog}(n))$ (as is the case if $m = O(\sqrt{d})$; cf. Lemma 8.17). Then $K[V]^G$ has a separating quasi-e.s.o.p. Furthermore, such a quasi-e.s.o.p. can be constructed by a uniform AC^0 circuit of quasi-polynomial size with oracle access to DET .*

Here DET denotes the determinant function.

For general m we have the following result which generalizes Theorem 1.1 (a)-(d) (1) to any finite dimensional rational representation V of G assuming the black-box derandomization hypothesis for general PIT (instead of STIT) and an additional hypothesis of about explicitness.

Theorem 1.4 *Suppose K is an algebraically closed field of characteristic zero. Let V as in (2) be a rational representation of $G = SL_m(K)$ of dimension n . Suppose the variety $V/G = \text{spec}(K[V]^G)$ is explicit in a relaxed sense (cf. Definition 8.9). Then:*

- (a) *A separating s.s.o.p. for $K[V]^G$ exists in a relaxed sense and can be constructed in $\text{poly}(n)$ work-space.*
- (b) *Suppose the black-box derandomization hypothesis for PIT over K holds. Then $K[V]^G$ has a separating e.s.o.p. in a relaxed sense.*
- (c) *Analogue of Theorem 1.1 (c) holds in this context.*
- (d) *Analogue of Theorem 1.1 (d) (1) also holds.*

This result also holds for any finite dimensional representation of a classical simple algebraic group in characteristic zero; cf. Theorem 9.1.

The general ring $K[V]^G$ in Theorem 1.4 is a basic prototype of the ring that arises in the context of the fundamental classification (moduli) problem [MFK] of algebraic geometry. Its finite generation is the celebrated result of Hilbert [H12]. Before this result it was not even known if a finite S , let alone a small S , such that $K[V]^G$ is integral over the subring generated by S , exists. Hilbert’s first proof of finite generation was nonconstructive. This was severely criticized by Gordan as “theology and not mathematics”. Noether’s Normalization Lemma as well as the Nullstellansatz were proved by Hilbert in the course of his second constructive proof of this result in response to this criticism. It is fair to say that this constructive proof and its various mathematical ingredients such as the Normalization Lemma and the Nullstellansatz changed the course of algebraic geometry, representation theory, and the theory of computation in the twentieth century. But Hilbert could only show that his algorithm for constructing finitely many generators for $K[V]^G$ worked in finite time. He could not prove any explicit upper bound on its running time. Such a bound was proved in [P] a century later, and improved significantly in [D]. This improved analysis, in conjunction with a recent fundamental advance [MR2] in Gröbner basis theory, yields an EXPSPACE-algorithm to compute a small S such that $K[V]^G$ is integral over the subring generated by S ; cf. Proposition 8.2. This algorithm needs exponential space and double exponential time even when m is constant, because the dimension of the ambient space containing V/G is exponential in n even when m is constant; cf. Section 2.4 and the remark after Proposition 8.2.

Theorem 1.3 shows that this double exponential time bound can be brought down to quasi-polynomial, thereby bringing NNL (even in the strong form) from *EXPSPACE* to quasi-*P* (in fact, quasi-*DET*) unconditionally if m is constant or $O(\text{polylog}(n))$. Classical invariant theory mainly focused on the case when m is constant. For example, Hilbert’s paper [H12] mainly focused on the case when $m = 4$. The problem of constructing a finite set of generators for $K[V]^G$ was not known to be decidable before this paper even in this case. Thus Theorem 1.3 does put NNL in quasi-DET unconditionally in the case of that Hilbert’s paper focused on.

Theorem 1.4 says that NNL (in a strong form) can be brought down from *EXPSPACE*, where it is currently, to *P* for any m assuming black-box derandomization of PIT and also assuming that the variety $V/G = \text{spec}(K[V]^G)$ is explicit in a relaxed sense (Conjecture 8.10) as per the general definition of an explicit variety (Definition 10.2) introduced in this article.

1.3 Equivalence

A large class of algebraic varieties that are studied in algebraic geometry are explicit. For arbitrary explicit varieties we have the following result.

Let the base field K be an algebraically closed field of characteristic zero.

Theorem 1.5 *For any explicit variety specified succinctly (Definition 10.2), the problem of derandomizing Noether’s Normalization Lemma over K is (1) in PSPACE unconditionally, (2) in $\Sigma_3 \subseteq PH$ assuming GRH, and (3) in P assuming a strengthened form of black-box derandomization of PIT (defined in Section 10.1).*

The following result says that this strengthened form of black-box derandomization of PIT is, in fact, equivalent to derandomization of Noether’s Normalization Lemma in a strict form

for a specific explicit variety as follows.

Theorem 1.6 (*Equivalence*)

- (1) A strengthened form (defined in Section 10.1) of black-box derandomization of symbolic determinant or trace identity testing (SDIT or STIT) is equivalent to derandomization of Noether's Normalization Lemma (in a certain strict form) for the explicit algebraic variety $\Delta[\det, m]$ associated with the determinant in [MS1] in the context of the permanent vs. determinant problem.
- (2) A strengthened form of black-box derandomization of general PIT over K is equivalent to derandomization of Noether's Normalization Lemma (in a strict form) for the explicit algebraic variety similarly associated with the complexity class P in [MS1] in the context of the algebraic P vs. NP problem.
- (3) A strengthened form of black-box derandomization of PIT for depth three circuits over K is equivalent to derandomization of Noether's Normalization Lemma (in a strict form) for the (polynomially high order) secant variety of the Chow variety.

See Section 10.4 for the full statements of these results. The explicit varieties occurring in this result are harder than the categorical quotient V/G in Theorem 1.1 in the following sense.

First, the variety V/G is normal and has rational singularities [Bt] (even in the general setting of Theorem 1.4), whereas $\Delta[\det, m]$ is not even normal [Ku].

Second, explicit defining equations are known for V/G in Theorem 1.1. They are given by the second fundamental theorem (SFT) for matrix invariants [Pr, Rz, Fo]. (For this reason, the problem of constructing an e.s.o.p. for $K[V]^G$ in Theorem 1.1 without requiring the separation property is expected to be easier.) Such defining (or close ² to defining) equations play a crucial role in the known cases of derandomization of Noether's Normalization Lemma that lead to black-box derandomization of nontrivial subclasses of PIT. For example, the explicit determinantal equations [L1] of the secant variety of the Veronese variety are crucial for derandomization of Noether's Normalization Lemma for this variety and black-box (quasi) derandomization of diagonal depth three circuits. They are implicitly used in all existing techniques for black-box derandomization of diagonal depth three circuits, such as [SV] and [ASS]. The explicit determinantal equations of the Grassmannian, G/P , and Schubert varieties are crucial for derandomization of Noether's Normalization Lemma for these varieties and black-box derandomization of the Grassmannian SDIT; cf. Section 11.2. They are implicitly used in all existing techniques for black-box derandomization of the Grassmannian SDIT, such as the one in Section 11.2 here or [FS].

But for the varieties in Theorem 1.6, finding explicit defining equations is a huge challenge. It is intimately related to the century-old plethysm and Kronecker problems of invariant theory; cf. [MS2, L1]. Finding explicit defining equations is also extremely hard for the explicit varieties that arise in the construction of a separating e.s.o.p. in Theorem 1.1 (b) assuming black-box derandomization; cf. Example (1) in Section 10.2.1. For V/G in Theorem 1.4, the problem of finding explicit defining equations, also called the problem of proving the second fundamental theorem (SFT) in general, is considered “completely unsolvable” (cf. page 238 in [PV]). Thus

²Strictly speaking, we do not need the equations to be defining in the context of NNL. We only need that they cut out a variety of polynomial dimension containing the variety under consideration.

a common feature of all wild problems considered in this paper is that the problem of finding explicit defining (or close to defining) equations for the underlying varieties turns out to be extremely hard. This seems to be the common cause of their wildness.

1.4 The GCT chasm

By Theorem 1.5, NNL for explicit varieties is in PH assuming GRH. Thus currently the main obstacle to putting NNL for general explicit varieties in PH unconditionally is GRH. Removing this obstacle already seems to be a huge challenge. As already mentioned after Theorem 1.1, GRH may suffice to put NNL in MA but possibly not NP . Bringing NNL from $PSPACE$ all the way to P , where the black-box derandomization hypothesis takes us (cf. Theorems 1.1, 1.5 and 1.6), may be much harder and well beyond the reach of the existing techniques even for depth three circuits.

Theorems 1.1, 1.2, and 1.6 may thus explain why proving black-box derandomization results for PIT for even depth three circuits, proving superpolynomial constant-depth-arithmetic-circuit or Boolean- TC^0 -circuit lower bounds for even EXP^{NP} , or proving even very conservative uniform Boolean conjectures such as $EXP^{NP} \not\subseteq \Sigma_2^P \cap \Pi_2^P$ has turned out to be so hard. In contrast, there are already known derandomization results for several versions of restricted PIT's and restricted classes of depth-three circuits in characteristic zero (cf. the survey [SY] and the references therein), a quadratic lower bound for depth three circuits in characteristic zero [SW], a quadratic determinantal lower bound for the permanent [MR], superpolynomial AC^0 lower bounds for parity and majority (cf. [BSi] and the references therein), a superpolynomial ACC lower bound for EXP^{NP} [Wi], and uniform hierarchy theorems such as $EXP^{NP} \not\subseteq \Delta_2^P$, along with the known barriers [BGS, RR, AW] to the proof techniques of some of these lower bounds. Since non-black-box derandomization of PIT is closely related to circuit lower bounds for $NEXP$ [KI], Theorem 1.2 (f) may explain why even partial non-black-box derandomization of PIT has turned out to be so hard. Theorems 1.1, 1.4, 1.5, and 1.6 may explain the difficulty of the classification problem in algebraic geometry from the complexity-theoretic perspective.

The results in this paper thus reveal a chasm, in the terminology of [AV], at depth three arithmetic circuits, at TC^0 Boolean circuits, and at uniform conjectures such as $EXP^{NP} \not\subseteq \Sigma_2^P \cap \Pi_2^P$. This chasm also extends to geometry (classification). As per Theorems 1.1, 1.2, 1.4, and 1.6, the root cause of this chasm common to geometry and complexity theory lies at the junction of these two fields, namely, the problem of derandomizing Noether's Normalization Lemma. We refer to the difficulty of derandomizing Noether's Normalization Lemma, or formally the existing $PSPACE$ (or rather ³ $EXPSPACE$) vs. P gap in the current knowledge, as the *GCT chasm*. The prefix GCT here refers to the location of the chasm at the junction of geometry and complexity theory. It does not mean that either geometry or complexity theory is necessary to cross it. Though both fields may turn out to be indispensable in practice.

We conjecture that (separating) e.s.o.p.'s exist for the $K[V]^G$'s under consideration and the coordinate rings of explicit varieties in general: i.e., Noether's Normalization Lemma for these rings can be derandomized (in a strong form), as suggested by Theorems 1.1, 1.2, 1.4, 1.5 and

³This is because strict NNL for V/G in Theorem 1.4, which is currently in $EXPSPACE$ for general m , is easier than (i.e. can be reduced to) the strict NNL for the explicit variety in Theorem 1.6 (2) assuming that V/G for general m is explicit in a relaxed sense (Conjecture 8.10).

1.6, cf. Conjecture 11.1, and the GCT chasm can be crossed.

By *geometric complexity theory* (GCT), we henceforth mean broadly any approach to cross this chasm based on a synthesis of geometry and complexity theory in some form. One plausible approach is suggested in this article (Section 11) and its sequel [Mu5] based on the constructions in the earlier papers [MS1, MS2, MNS, BMS] in this series.

1.5 Proof technique

The proof of Theorem 1.1 (a), (b), and (c) is based on the fundamental works in geometric invariant theory and algebraic complexity theory, specifically, the first and second fundamental theorems (FFT and SFT) for matrix invariants in [Pr, Rz], the properties of the categorical quotient $V/G = \text{spec}(K[V]^G)$ proved in [MFK, DK], and the existence of a hitting set for PIT proved in [HS]. The key idea in the proof of Theorem 1.1 (a) is to show, using these FFT and SFT, that the categorical quotient $V/G = \text{spec}(K[V]^G)$ that arises therein is an *explicit* variety (cf. Lemma 3.7). Specifically, we show that, given m and r , one can construct in $\text{poly}(n)$ time symbolic trace polynomials $F_j(v, y)$, $1 \leq j \leq m^2$, over K , where $v = (v_1, \dots, v_n)$ are the coordinates of V and $y = (y_1, \dots, y_n)$ are auxiliary variables, such that $F_j(v, y)$'s can be expressed as linear combinations of linearly independent symbolic traces over y with coefficients in $\mathbb{Q}[v]$ that generate $K[V]^G$. Specializing the y -variables of $F_j(v, y)$'s at the elements of any hitting set T for STIT, which exists by [HS], yields a small separating set $S \subseteq K[V]^G$ of invariants with small specifications in the form of symbolic traces. It follows, by [MFK, DK], that S is a separating s.s.o.p.. This proof can be made constructive so as to construct an S in $\text{poly}(n)$ work-space using the PSPACE-algorithm [Ko, Ko1] for Hilbert's Nullstellansatz. This proves Theorem 1.1 (a). The proof of Theorem 1.1 (b) (1) is based on the properties of the categorical quotient V/G proved in [MFK, DK]. If the hitting set T in the proof of Theorem 1.1 (a) is chosen explicitly, as can be done if the black-box derandomization hypothesis for STIT holds, then S can be constructed in $\text{poly}(n)$ time. In this case S is a separating e.s.o.p. This proves Theorem 1.1 (b) (2). The proof of Theorem 1.1 (c) is a variation of this idea. Theorem 1.1 (d) (1) follows if one uses the fundamental PH-algorithm for Hilbert's Nullstellansatz in [Ko1] in place of the PSPACE-algorithm used in the proof of Theorem 1.1 (a).

Theorem 1.1 (e) and (f) is based on the characterization [Ar] of the closed G -orbits in V and the polynomial time algorithms [FR, Eb] for computing the decomposition of a finite-dimensional algebra.

Theorem 1.2 (a) and (b) are reduced to Theorem 1.1 (b) and (c), respectively, using the fundamental equivalence between black-box derandomization of PIT and circuit lower bounds for EXP [HS, IW, KI, Ag], efficient factorization of multivariate polynomials given by straight-line programs [KI], and the depth reduction for arithmetic circuits [VSB, AV, Ko2]. Theorem 1.2 (c) and (d) are reduced to Theorem 1.2 (a) and (b) using the fundamental connection based on GRH between Boolean and algebraic complexities [Bu, Ko1], and the TC^0 -algorithms for division and iterated multiplication [HAB]. Theorem 1.2 (e) is reduced to Theorem 1.2 (d) using the the fundamental connection between nonuniform and uniform Boolean lower bounds [KL, BFNW, IKW]. Theorem 1.2 (f) is obtained by extending the proof of Theorem 1.2 (a)-(e), plugging the NP -oracle in the right places.

A similar extension of the proof of Theorem 1.2 (c), plugging the NP^{NP} -oracle in the right

places, yields a quasi-e.s.o.p. S for $K[V]^G$ giving the algorithm for constructing S an access to the NP^{NP} oracle, and assuming (i) GRH , and (ii) a subexponential circuit size lower bound for Δ_3^{EXP} . But (ii) holds unconditionally by the fundamental subexponential circuit size lower bound for Δ_3^{EXP} [Kn, MVW]. This implies that S is a quasi-e.s.o.p., with an access to the NP^{NP} oracle, assuming GRH alone. This proves Theorem 1.1 (d) (2). To prove Theorem 1.1 (d) (3), one uses the fundamental circuit lower bound for MA_{EXP} [MVW] instead in the last step.

The proof of Theorem 1.4 is obtained by generalizing the proof technique of Theorems 1.1 and 1.2, in conjunction with the algorithm in [H12] for constructing finitely many generators for $K[V]^G$, computational invariant theory [Stm2, D], the bounds for the degrees of the generators for $K[V]^G$ in [P, D], and standard monomial theory [LR, DRS].

The proof of Theorem 1.3 differs from the proof of Theorem 1.4 in two places. First, explicitness of $V/G = \text{spec}(K[V]^G)$ for constant m can be proved unconditionally (Theorem 8.8) using properties of the Reynolds operator [DK, Stm2], standard monomial theory [LR, DRS], and algebraic complexity theory [Cs, MP, Str1, Str2]. This is the heart of the proof. Second, one only needs the black-box (quasi) derandomization hypothesis for diagonal depth three circuits [Sx] in this case. This hypothesis is a consequence of the proof technique in [SV] (cf. Theorem 2.4), and is also proved in [ASS]. Hence quasi-derandomization of Noether's Normalization Lemma follows unconditionally when m is constant or $O(\text{polylog}(n))$. This proves Theorem 1.3.

Theorems 1.5 and 1.6 are proved by abstracting the ideas in the proofs of Theorems 1.1, 1.3 and 1.4.

1.6 Organization

The rest of this article is organized as follows.

In Section 2 we recall some results and notions in complexity theory and geometric invariant theory that we need in this paper. Theorem 1.1 (a), (b), (c), (d) (1), (e), and (f) are proved in Section 3. These results (except for (e) and (f)) are generalized to arbitrary quivers in Section 4. Theorem 1.2 (a), (b), and a half of (f) are proved in Section 5. Theorem 1.2 (c)-(d), and the other half of (f) are proved in Section 6. All the ingredients in the preceding sections are put together to prove Theorem 1.1 (d) (2) and (3) in Section 7. Theorems 1.3 and 1.4 are proved in Section 8. Generalization of Theorems 1.3 and 1.4 to classical simple algebraic groups are proved in Section 9. Theorems 1.5 and 1.6 are proved in Section 10. One plausible approach to derandomization of PIT based on the equivalence in this article is formulated in Section 11.

Acknowledgment: In the first draft of this paper, Theorem 1.3 was stated and proved conditionally, assuming the black-box derandomization hypothesis for diagonal depth three circuits [Sx]. The author is grateful to Amir Shpilka and Michael Forbes for pointing out that this hypothesis (allowing a quasi prefix) is a consequence (cf. Theorem 2.4) of the fundamental black-box derandomization technique in [SV] for the read-once PIT and that it is also proved in [ASS]. Hence, Theorem 1.3 holds unconditionally. The author is also grateful to Sanjeev Arora, Jonah Blasiak, Josh Grochow, and Joseph Landsberg for helpful comments.

2 Preliminaries

In this section we recall some results and notions in complexity theory and geometric invariant theory that we need in this paper. We assume familiarity with basic complexity theory [AB], algebraic geometry [Mm], and representation theory [Fu].

2.1 Black-box derandomization hypothesis

We begin by stating in more detail the black-box derandomization hypothesis for polynomial identity testing [HS, IW, KI, Ag, DSY] that we need.

The polynomial identity testing (PIT) problem over a field K is the problem of deciding if a given arithmetic circuit $C(x)$, $x = (x_1, \dots, x_r)$, over K of size at most s computes an identically zero polynomial. In this paper, the size of the circuit means the total number of edges in it. There is no restriction on the bit-lengths of the constants in the circuit. We also assume, unless stated otherwise, that K is an algebraically closed field of characteristic zero. By the PIT problem of small degree, we mean the PIT problem wherein the polynomial computed by the circuit is assumed to have a small $O(s^c)$ degree, for some fixed constant $c > 0$. The article [IM] gives a randomized algorithm to solve the PIT problem in $\text{poly}(s)$ operations over K . This is a black-box algorithm in the sense that it does not look inside the circuit. It merely evaluates the circuit at randomly chosen test inputs.

The black-box derandomization problem for PIT [HS, IW, KI, Ag] is to design an efficient deterministic black-box algorithm for solving the PIT problem. Specifically, the problem is to construct efficiently a *hitting set* against all circuits over K with size $\leq s$ and on $r \leq s$ variables. By a hitting set, we mean a set $S_{r,s} \subseteq \mathbb{N}^r$ of test inputs such that (1) the bit-length of the specification of each test input is $\text{poly}(s)$, and (2) for every circuit C on K and r variables with size $\leq s$ computing a non-zero polynomial $C(x)$, $S_{r,s}$ contains a test input b such that $C(b) \neq 0$. The *black-box-derandomization hypothesis* [HS, IW, KI, Ag, DSY] in this context is that there exists a hitting set of $\text{poly}(s)$ total bit-size that is computable in $\text{poly}(s)$ time. More generally, if a hitting set has $O(T(s))$ size and is computable in $O(T(s))$ time, we say that PIT for circuits has $T(s)$ -time-computable black-box derandomization. By our definition of a hitting set, it is still required here that the bit-length of each test input in the hitting set be $\text{poly}(s)$.

We have also defined a restricted form of PIT called symbolic trace identity testing (STIT) in Section 1.1. It is equivalent [Be, MP, Sp], up to polynomial factors, to symbolic determinant identity testing (SDIT) defined as follows. Let Y be a variable $m \times m$ matrix. Let Y' be any $m \times m$ matrix, whose each entry is a homogeneous linear form over K in the variable entries y_{ij} 's of Y . We call $\det(Y')$ a symbolic determinant of size m . By SDIT, we mean the problem of deciding, given Y' , if the symbolic determinant $\det(Y')$ is an identically zero polynomial in y_{ij} 's. The black-box derandomization hypothesis for SDIT is that, given m , one can construct in $\text{poly}(m)$ time a hitting set against all non-zero symbolic determinants over K of size m . This is equivalent to the black-box derandomization hypothesis for STIT in Section 1.1. The parallel black-box derandomization hypothesis for SDIT is that a hitting set is computable by a uniform AC^0 circuit of $\text{poly}(m)$ bit-size with oracle access to DET (the determinant function). The equivalent parallel black-box derandomization hypothesis for STIT is that a hitting set against symbolic traces (of degree $\leq m$) over K for $m \times m$ matrices and r variables is computable by a

uniform AC^0 circuit of $\text{poly}(m, r)$ bit-size with oracle access to DET .

The following result says that a hitting set for PIT exists.

Theorem 2.1 (Heintz, Schnorr) (cf. Theorem 4.4 in [HS]) *Let K be any field of characteristic zero. There exists a hitting set $B \subseteq [u]^r$, $u = 2s(d+1)^2$, of size $6(s+1+r)^2$ against all arithmetic circuits over K and r variables of size $\leq s$ and degree $\leq d$.*

The proof of this result in [HS] does not yield any efficient algorithm for constructing B .

The following result is a variant of Theorem 7.7 in [KI]. This is why PIT is expected to have efficient black-box derandomization.

Theorem 2.2 (Kabanets and Impagliazzo) (cf. Theorem 7.7 in [KI]) *Suppose K is a field of characteristic zero.*

(a) *Suppose there is an exponential-time computable multilinear polynomial $f(x_1, \dots, x_m)$ with integral coefficients of $\text{poly}(m)$ bit-length such that f can not be evaluated by an arithmetic circuit over K of $O(2^{m^a})$ size for some constant $a > 0$. Then PIT for small degree circuits of size $\leq s$ has $O(2^{\text{polylog}(s)})$ -time computable black-box derandomization.*

(b) *If f cannot be evaluated by an arithmetic circuit over K of $O(m^a)$ size for any constant $a > 0$, $m \rightarrow \infty$, then PIT for small degree circuits of size $\leq s$ has $O(2^{s^\epsilon})$ -time computable black-box derandomization, for any $\epsilon > 0$.*

The proof of this result is very similar to that of Theorem 7.7 in [KI] (which works in the black-box model). Hence we omit its details and only point out how to take care of the main difference between the setting in [KI] and the one here. The difference is that in [KI] the size of the circuit is defined to be the total number of edges in it plus the total bit-length of the constants in it, whereas here the size just means the total number of edges. A key ingredient in the proof in [KI] is an efficient algorithm in [KI] for factoring multivariate polynomials (cf. Lemma 7.6. in [KI]). In its place we use instead the following result in [Kl, KT] that does not depend on the bit-lengths of the constants in the circuit.

Theorem 2.3 (Kaltofen) (cf. Corollary 6.2. in [Kl] and Theorem 1 in [KT])

Let K be a field of characteristic zero. Suppose $g(x_1, \dots, x_n)$ is p -computable [V1] over K . This means g is a polynomial of $\text{poly}(n)$ degree that can be computed by a nonuniform circuit over K of $\text{poly}(n)$ size. Then each factor of g in $K[x_1, \dots, x_n]$ is also p -computable over K .

More generally, given any polynomial $g \in K[x_1, \dots, x_n]$ and a polynomial $f \in K[x_1, \dots, x_n]$ dividing g , there exists a nonuniform circuit over K of $\text{poly}(n, \deg(g))$ size, with oracle gates for g , that computes f .

For the converse of Theorem 2.2, see [HS, Ag, SW].

For the proof of Theorem 1.3, we will need a restricted form of PIT for diagonal depth three circuits [Sx]. By a diagonal depth three circuit, we mean a circuit $C(x)$, $x = (x_1, \dots, x_r)$, that computes a sum of powers of linear functions:

$$C(x) = \sum_{i=1}^k l_i^{e_i},$$

where each l_i is a possibly non-homogeneous linear form in x_i 's with coefficients in K . Here k is called the top fan-in of the circuit, and $e = \max\{e_i\}$ its degree. The size of this circuit is $s = O(rek)$.

The black-box derandomization hypothesis in this context is that a hitting set against diagonal depth three circuits on r variables with degree $\leq e$ and top fan-in $\leq k$ can be computed in $\text{poly}(s)$ time. The parallel black-box derandomization hypothesis is that such a hitting set can be computed by a uniform AC^0 circuit of $\text{poly}(s)$ bit-size. This holds unconditionally allowing a quasi-prefix:

Theorem 2.4 (Shpilka, Volkovich; Agrawal, Saha, Saxena) *(cf. Theorem 6.4 in [SV] and appendix in [ASS]) A hitting set against all diagonal depth three circuits over K of size $\leq s$ can be constructed by a uniform AC^0 circuit of $O(s^{O(\log s)})$ size.*

This result is a variant of Theorem 6.4 in [SV], with essentially the same proof [Sp]. Specifically, the partial derivative method in [SV] (also cf. [RSh]) implies that if $2^l > ke$ then no diagonal depth three circuit over x_1, \dots, x_r with degree $\leq e$ and top fan-in $\leq k$ can compute a polynomial of the form $x_{i_1} \cdots x_{i_l} f(x_1, \dots, x_r)$, for distinct i_j 's. This lower bound, in conjunction with the proof technique of Theorem 6.4 in [SV], implies that the set of r -vectors with entries in $0, \dots, e$ that contain at most l nonzero entries is a hitting set against diagonal depth three circuits over x_1, \dots, x_r with degree $\leq e$ and top fan-in $\leq k$.

2.2 Geometric invariant theory

We now state some results from geometric invariant theory and their consequences that are needed for proving Theorem 1.1.

Let K be an algebraically closed field of characteristic zero, $M_m(K)$ the space of $m \times m$ matrices over K , and $V = M_m(K)^r$, the direct sum of r copies of $M_m(K)$. Let $n = \dim(V) = rm^2$. The space V has the adjoint action of $G = SL_m(K)$:

$$(A_1, \dots, A_r) \rightarrow (PA_1P^{-1}, \dots, PA_rP^{-1}), \quad (3)$$

where $A_1, \dots, A_r \in M_m(K)$ and $P \in SL_m(K)$. Let U_1, \dots, U_r be variable $m \times m$ matrices. Then the coordinate ring $K[V]$ of V can be identified with the ring $K[U_1, \dots, U_r]$ generated by the variable entries of U_i 's. Let $K[V]^G \subseteq K[V]$ be the ring of invariants with respect to the adjoint action.

Theorem 2.5 (Procesi-Razmyslov-Formanek) *[Pr, Rz, Fo] (The First Fundamental Theorem (FFT) of matrix invariants; cf. Theorems 6 and 10 in [Fo]) The ring $K[V]^G$ is generated by traces of the form $\text{trace}(U_{i_1} \cdots U_{i_l})$, $l \leq m^2$, $i_1, \dots, i_l \in [r] = \{1, \dots, r\}$.*

Furthermore, $K[V]^G$ is not generated by traces of degree $\leq m^2/8$ (with $r = m^2$).

Let $K[S_r]$ be the group algebra of the symmetric group S_r on r letters. Write any $\sigma \in S_r$ as a product of disjoint cycles:

$$\sigma = (a_1 \cdots a_{k_1})(b_1 \cdots b_{k_2}) \dots,$$

where 1-cycles are included, so that each of the numbers $1, \dots, r$ occurs exactly once. Define

$$T_\sigma(U_1, \dots, U_r) = T(U_{a_1} \cdots U_{a_{k_1}}) T(U_{b_1} \cdots U_{b_{k_2}}) \cdots. \quad (4)$$

Theorem 2.6 (Procesi-Razmyslov) [Pr, Rz] *(The Second Fundamental Theorem (SFT) of matrix invariants; cf. Theorem 1 in [Fo])* Let $J(m, r)$ be the two-sided ideal of KS_r which is the sum of all simple factors of KS_r corresponding to the Young diagrams with $\geq m + 1$ rows. Define the K -linear map $\phi : KS_r \rightarrow K[V]^G$ by

$$\phi\left(\sum a_\sigma \sigma\right) = \sum a_\sigma T_\sigma(U_1, \dots, U_r).$$

Then $\text{Ker}(\phi) = J(m, r)$. Furthermore, $J(m, r) = 0$ if $r \leq m$.

Let X_1, \dots, X_r be $k \times k$ variable matrices. For any word $\alpha = i_1, \dots, i_l$, $l \leq k$, $i_j \in [r] = \{1, \dots, r\}$, let

$$T_\alpha(X) = \text{trace}(X_{i_1} \cdots X_{i_l}), \quad (5)$$

where $X = (X_1, \dots, X_r)$. We say that two words α and α' are equivalent, if α' can be obtained from α by circular rotation. In this case, $T_\alpha(X) = T_{\alpha'}(X)$. Let $[\alpha]$ denote the equivalence class of words equivalent to α . Let $T_{[\alpha]}(X) = T_\alpha(X)$; the choice of α in $[\alpha]$ does not matter.

Corollary 2.7 *The traces $\{T_{[\alpha]}(X)\}$, where $[\alpha]$ ranges over all equivalence classes of words of length $l \leq k$, are linearly independent.*

Proof: Suppose to the contrary that there is a linear dependence

$$\sum_{[\alpha]} b_{[\alpha]} T_{[\alpha]}(X) = 0, \quad b_{[\alpha]} \in K. \quad (6)$$

Without loss of generality, we can assume that this relation is homogeneous in X_i 's. We can also assume that it is multilinear in X_i 's. Otherwise, we can multilinearize it by substituting

$$X_i = \sum_{j=1}^{d_i} t_i^j X_i^j,$$

in the l.h.s. of (6) and equating the coefficient of $\prod_i \prod_{j=1}^{d_i} t_i^j$ to zero, where d_i is the degree of X_i in the relation, t_i^j 's are new variables, and X_i^j 's are new variable $k \times k$ matrices.

If the dependence is multilinear and homogeneous, then each $[\alpha] = [i_1, \dots, i_l]$ corresponds to the permutation in S_l with just one cycle (i_1, \dots, i_l) . We denote it by $[\alpha]$ again. Applying Theorem 2.6 with $U = X$, $r = l$, and $m = k$, it follows that $b_{[\alpha]}$'s are all zero, since $J(k, l) = 0$ for $l \leq k$. Q.E.D.

The following is an alternative form of Theorem 2.6.

The Cayley-Hamilton theorem implies [Pr] the fundamental trace identity

$$F(U_1, \dots, U_{m+1}) = \sum_{\sigma \in S_{m+1}} \text{sign}(\sigma) T_\sigma(U_1, \dots, U_{m+1}) = 0,$$

where the trace function T_σ is defined as in (4).

Theorem 2.8 (Procesi-Razmyslov) (cf. Theorem 4.5 in [Pr]) *The ideal of all relations among the trace monomial generators of $K[V]^G$ given by Theorem 2.5 is generated by the elements of the form $F(M_1, \dots, M_{m+1})$, where M_i 's range over all possible monomials in U_j 's so that the total length of M_i 's is $\leq m^2$.*

This follows from the proof of Theorem 4.5 in [Pr].

We state the next results in more generality than needed for the proof of Theorem 1.1, since we need them in this generality for the proofs of Theorems 1.3, 1.4, and 9.1.

So assume now that G is any reductive algebraic group defined over an algebraically closed field K of characteristic zero and V its any finite dimensional rational representation. Then the invariant ring $K[V]^G$ is finitely generated [H12]. So we can consider the variety $V/G = \text{spec}(K[V]^G)$, called the categorical quotient [MFK]. It has the following property (Theorem 2.9) that plays a crucial role in the proof of Theorem 1.1 as well as other results (Theorems 1.3 and 1.4) in this paper.

Fix any set $F = \{f_1, \dots, f_t\}$ of non-constant homogeneous generators of $K[V]^G$. Consider the morphism $\pi_{V/G}$ from V to K^t given by

$$\pi_{V/G} : v \rightarrow (f_1(v), \dots, f_t(v)). \quad (7)$$

Then V/G can be identified with the closure of the image of this morphism. Let $z = (z_1, \dots, z_t)$ be the coordinates of K^t , I the ideal of V/G under this embedding, $K[V/G]$ its coordinate ring. Then $K[V/G] = K[z]/I$, and we have the comorphism $\pi_{V/G}^* : K[V/G] \rightarrow K[V]$ given by

$$\pi_{V/G}^*(z_i) = f_i. \quad (8)$$

Since f_i are homogeneous, $K[V/G]$ is a graded ring, with the grading given by $\deg(z_i) = \deg(f_i)$. Furthermore, $\pi_{V/G}^*$ gives the isomorphism between $K[V/G]$ and $K[V]^G$, and we have $\pi_{V/G}^*(K[V/G]) = K[V]^G$.

Theorem 2.9 (Mumford [MFK]) (cf. Theorem 1.1 in [MFK] and Theorem 4.6 and 4.7 in [PV])

- (a) *The image of $\pi_{V/G}$ is already closed. Hence the map $\pi_{V/G} : V \rightarrow V/G$ is surjective.*
- (b) *For any $x \in V/G$, $\pi_{V/G}^{-1}(x)$ contains a unique closed G -orbit.*
- (c) *For any G -invariant subvariety $W \subseteq V$, $\pi_{V/G}(W)$ is a closed subvariety of V/G .*
- (d) *Given $v, w \in V$, the closures of the G -orbits of v and w intersect iff $r(v) = r(w)$ for all $r \in K[V]^G$.*

The following is a graded variant of Noether's normalization Lemma implicit in its proof; cf. Theorem 13.3. in [DEP2], corollary 2.29 in [Mm], and also the proof of Theorem 1.5.17 in [BrH].

Lemma 2.10 (Graded Noether Normalization) (a) *Let f_1, \dots, f_t be any non-constant homogeneous generators of $K[V]^G$, and $H \subseteq K[V]^G$ any set of homogeneous elements such that,*

letting $I(H)$ denote the ideal generated by H , $f_i^{e_i} \in I(H)$ for some positive integer e_i , for every i . Then $K[V]^G$ is integral over the subring generated by H . Analogous result holds for any positively graded affine K -algebra R in place of $K[V]^G$.

(b) Let E be any set of homogeneous elements in $K[V/G]$ such that $(V/G) \cap Z(E) = \{0\}$. Here V/G is embedded in K^t as above, $Z(E) \subseteq K^t$ denotes the zero set of E , and 0 denotes the origin in K^t . Then $K[V/G]$ is integral over the subring generated by E .

If d_i 's are positive integers so that $g_1 = f_1^{d_1}, \dots, g_t = f_t^{d_t}$ have the same degree, then any H , $|H| \geq \dim(K[V]^G)$, consisting of random (generic) linear combinations of g_i 's has the property in Lemma 2.10. A set H of homogeneous invariants of cardinality equal to $\dim(K[V]^G)$ such that $K[V]^G$ is integral over the subring generated by H is called an h.s.o.p. (homogeneous system of parameters) of $K[V]^G$.

2.2.1 Separating invariants

Following Derksen and Kemper [DK] (cf. Section 2.3.2 therein), let us call a set $S \subseteq K[V]^G$ separating if for any $v, w \in V$ such that $r(v) \neq r(w)$, for some $r \in K[V]^G$, there exists an $s \in S$ such that $s(v) \neq s(w)$.

Theorem 2.11 (Derksen, Kemper) (cf. Theorem 2.3.12 in [DK]) *Let $S \subseteq K[V]^G$ be a finite separating set of homogeneous invariants. Then $K[V]^G$ is integral over the subring generated by S .*

Proposition 2.12 *Suppose $K[V]^G$ is generated by a set homogeneous invariants of degree $\leq D$. Then $K[V]^G$ has a separating set S of homogeneous invariants of cardinality $\leq 2nD$.*

Proof: Write $V^2 = V \times V$ for the product of two copies of V . Let $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ be the coordinates of the two copies. Let $F = \{f_1(v), \dots, f_t(v)\}$ be a set of non-constant homogeneous generators of $K[V]^G$ of degree $\leq D$. For any $1 \leq j \leq D$, let $R_j[V, G] \subseteq K[V^2]$ be the subring generated by $H_j = \{h_i(v, w) = f_i(v) - f_i(w) \mid \deg(f_i) = j\}$. Clearly $\dim(R_j[V, G]) \leq 2n$. Let $X_j[V, G] = \text{spec}(R_j[V, G])$. By Noether's Normalization Lemma [E] (Lemma 2.10), each $R_j[V, G]$ has an h.s.o.p. E_j , $|E_j| = \dim(R_j[V, G]) \leq 2n$, whose each element is a linear combination of h_i 's in H_j . Let $E = \cup_j E_j$. Each element of E is of the form $g(v) - g(w)$ for some $g(v)$ that is a linear combination of $f_i(v)$'s of the same degree. Let $S = \{g \mid g(v) - g(w) \in E\}$. Then S is separating. To see this, let $a, b \in V$ be two points such that $r(a) \neq r(b)$ for some $r \in K(V)^G$. Then $f_i(a) \neq f_i(b)$ for some $1 \leq i \leq t$. Suppose to the contrary, that $g(a) = g(b)$ for all $g \in S$. This means every element of E vanishes at (a, b) . Since $f_i(v) - f_i(w)$ is integral over the subring generated by E , it follows that $f_i(a) = f_i(b)$; a contradiction. Q.E.D.

2.3 GRH and solving polynomial equations

In this section we state the known implications of GRH in the context of solving polynomial equations and their consequences that we need in this paper.

Let $f_1, \dots, f_s \in \mathbb{Z}[x_1, \dots, x_n]$ be integral polynomials of degree $d \geq n$ and weight $\leq w$. Here the weight of a polynomial is defined to be the sum of the absolute values of its coefficients. Suppose the system of polynomial equations

$$(S) : f_1 = 0, \dots, f_s = 0$$

is solvable over \mathbb{C} . Let $\pi(x)$ denotes the number of primes $\leq x$ and $\pi_S(x)$ the number of primes $p \leq x$ such that (S) is solvable over the finite field F_p .

Theorem 2.13 (Bürgisser) (cf. Theorem 4.4. in [Bu]) Assuming GRH, $\pi_S(X) \geq \frac{\pi(x)}{d^{O(n)}} - x^{1/2} \log(wx)$ for all systems (S) solvable over \mathbb{C} .

Theorem 2.14 (Koiran: Hilbert's Nullstellensatz is in PH assuming GRH) (cf. [Ko1]) Assuming GRH, the problem of deciding if a given system of multivariate integral polynomials, specified as straight-line programs, has a complex solution belongs to $RP^{NP} \subseteq \Pi_2$.

This result is stated in [Ko1] for a sparse representation of polynomials. But it can be seen to hold for a representation of polynomials by straight-line programs as well.

Lemma 2.15 (Noether Normalization is in PH for the long representation assuming GRH)

Let K be an algebraically closed field of characteristic zero. Let $Z \subseteq K^t$ be the variety consisting of the common zeroes of a set of homogeneous integral polynomials $f_1(z), \dots, f_s(z)$, $z = (z_1, \dots, z_t)$. We assume that f_i 's are specified as straight-line programs. Let N denote the total bit-length of the specification of f_i 's. Let $\dim(Z)$ be the dimension of Z .

(a) Consider random linear forms $L_r(z) = \sum_k b_{k,r} z_k$, $0 \leq r \leq s$, where $b_{k,r}$'s are random integers of large enough $\text{poly}(N)$ bit-length. Let $H_r \subseteq K^t$ be the hyperplane defined by $L_r(z) = 0$. If $s < \dim(Z)$, then $Z \cap \bigcap_r H_r \neq \{0\}$. If $s = \dim(Z)$, then with high probability, $Z \cap \bigcap_r H_r = \{0\}$, which implies (Lemma 2.10) that the homogeneous coordinate ring of Z is integral over the subring generated by $L_r(z)$'s.

(b) The problem of computing the linear forms $L_r(z)$'s such that $Z \cap \bigcap_r H_r = \{0\}$ belongs to PSPACE. This means the specifications of such linear forms can be computed in $\text{poly}(N)$ work space.

(c) Assuming GRH, this problem is in $RP^{NP} \subseteq \Pi_2$.

Proof:

(a) Let us assume that $s = \dim(Z)$, the other case being easy. Let $d = \max\{\deg(f_i)\}$. Clearly, $d \leq 2^N$. By raising f_i 's to appropriate powers, we can assume that all of them have the same degree $D \leq 2^{N^2}$. Consider generic combinations of f_i 's and generic linear forms:

$$\begin{aligned} F_j(z) &= \sum_i y_{i,j} f_i(z), & 1 \leq j \leq t - \dim(Z), \\ L_r(z) &= \sum_k w_{k,r} z_k, & 1 \leq r \leq \dim(Z), \end{aligned} \tag{9}$$

where $y_{i,j}$'s and $w_{k,r}$'s are indeterminates. Let R denote the multivariate resultant of F_j 's and L_l 's. It is a polynomial in $y_{i,j}$'s and $w_{k,l}$'s of degree $\leq D^t$. By Noether's Normalization Lemma

(cf. Corollary 2.29 in [Mm] and Lemma 2.10), the system of equations (9) has only $\{0\}$ as its solution for some rational values for $y_{i,j}$'s and $w_{k,r}$'s. Hence R is not identically zero as a polynomial in $y_{i,j}$'s and $w_{k,r}$'s. By the Schwarz-Zippel lemma [Sc], we can specialize $y_{i,j}$'s randomly to some rational values of $O(\log(D^t)) = \text{poly}(N)$ bit-length so that the resulting specialization R' of R is not identically zero. Then R' is a nonzero polynomial in $w_{k,r}$'s of degree $\leq D^t$. By the Schwarz-Zippel lemma again, R' does not vanish identically if we let $w_{k,r} = b_{k,r}$ for randomly chosen integers of bit-length $O(\log(D^t)) = \text{poly}(N)$. For such $b_{k,r}$'s, $Z \cap \bigcap_r H_r = \{0\}$.

(b) and (c): First, let us assume that we know $\dim(Z)$. Let $s = \dim(Z)$. Choose $b_{k,r}$'s as above randomly of large enough $\text{poly}(N)$ bit-length and test if $Z \cap \bigcap_r H_r = \{0\}$. The latter test can be done in polynomial space since Hilbert's Nullstellensatz is in PSPACE [Ko, Ko1, Cn] (unconditionally): choose random $y_{i,j}$'s and test if the resultant R above, which can be computed in polynomial space [Cn], is zero. Randomization can be removed since $RPSPACE = NPSPACE = PSPACE$. Assuming GRH, this test can be done by an RP^{NP} -algorithm, since Hilbert's Nullstellensatz is then in RP^{NP} (Theorem 2.14).

The problem that remains is that we do not really know $\dim(Z)$ a priori. So we start with a guess s for $\dim(Z)$, starting with $s = 0$ and increasing it one by one. At each step we randomly choose large enough $b_{k,r}$'s and $y_{i,j}$'s and test if R is non-zero. As long as $s < \dim(Z)$, R will always be zero. We stop as soon as we find that R is not zero. Randomization can be removed if we only want a PSPACE-algorithm. Q.E.D.

2.4 Succinct vs. long representation

Lemma 2.15 shows that the problem of constructing an h.s.o.p. for the coordinate ring of a variety Z is in PSPACE unconditionally and in PH assuming GRH if Z is specified by writing down a set of generators for its ideal—we call this a *long representation* of Z . It does *not* show that this problem is in PSPACE or PH (assuming GRH) for the succinct representation of explicit varieties used in this paper. For example, V/G in Theorem 1.1 is specified by just giving m and r in unary, and V/G in Theorem 1.4 is specified by just giving n and m (in unary) and the multiplicities (in unary) of the various Weyl modules $V_\lambda(G)$'s in the decomposition of V . The bit-lengths of the long representations of these varieties are exponential in the bit-lengths of their succinct representations. This is because the dimension t of the ambient space K^t containing V/G is exponential in the bit-length of the succinct representation.

Hence, Lemma 2.15 can only be used (Proposition 3.1) to show that the problem of constructing an h.s.o.p. for $K[V]^G$ in Theorem 1.1 is in EXPSPACE unconditionally and in the *exponential* hierarchy (*not* polynomial) assuming GRH. Similar result also holds for any variety for which explicit defining equations are known akin to the ones provided for V/G in Theorem 1.1 by the second fundamental theorem for matrix invariants [Pr, Rz]; cf. Theorem 10.7 (e).

For the varieties for which such explicit defining equations are not known, e.g. V/G in Theorem 1.3 or 1.4, Lemma 2.15 can only be used to show that the problem of constructing an h.s.o.p. is in EXPSPACE; cf. Proposition 8.2. This is because for such varieties the conversion of the succinct representation into a long representation itself takes exponential space (in the bit-length of the succinct representation) and double exponential time at present; cf. Lemma 8.3. In a nutshell, this is why the existing techniques for Noether normalization applied to V/G in Theorem 1.4 take exponential space and double exponential time even when m is constant; cf.

Proposition 8.2. The existing techniques also take exponential space and double exponential time for constructing a small separating set for $K[V]^G$ for similar reasons; cf. Proposition 3.2.

3 Implications of black-box derandomization of STIT

In this section we prove Theorem 1.1 (a), (b), (c), (d) (1), (e), (f).

Let K be an algebraically closed field of characteristic zero. Let $V = M_m(K)^r$, with the adjoint action of $G = SL_m(K)$, be as in Theorem 1.1. Let $n = \dim(V) = rm^2$. Let U_1, \dots, U_r be variable $m \times m$ matrices. Identify the coordinate ring $K[V]$ of V with the ring $K[U_1, \dots, U_r]$ generated by the variable entries of U_i 's. Let $K[V]^G \subseteq K[V]$ be the ring of invariants with respect to the adjoint action. We shall specify V and G succinctly by the pair (m, r) in unary. The goal is to construct a small separating set $S \subseteq K[V]^G$ of homogeneous invariants efficiently.

For any word $\alpha = i_1, \dots, i_l$, $l \leq m^2$, $i_j \in [r]$, let

$$T_\alpha(U) = \text{trace}(U_{i_1} \cdots U_{i_l}), \quad (10)$$

where $U = (U_1, \dots, U_r)$. Let

$$F = \{T_{[\alpha]}(U)\}, \quad (11)$$

where $[\alpha]$ ranges over the equivalence classes (under circular rotation) of all words in $1, \dots, r$ of length $\leq m^2$. Then F generates $K[V]^G$ by Theorem 2.5.

Let $t = |F|$ and let $z_{[\alpha]}$'s denote the coordinates of K^t . Consider the map $\pi_{V/G} : V \rightarrow K^t$,

$$\pi_{V/G} : A = (A_1, \dots, A_r) \rightarrow (\dots, T_{[\alpha]}(A), \dots). \quad (12)$$

Then V/G is the closure of the image of this map. By Theorem 2.9, this image is already closed. Hence $V/G = \text{Im}(\pi_{V/G}) = \overline{\text{Im}(\pi_{V/G})}$. Furthermore, $\pi_{V/G}^*(z_{[\alpha]}) = T_{[\alpha]}(U)$ and $\deg(z_{[\alpha]}) = \deg(T_{[\alpha]}(U))$.

3.1 Construction of an h.s.o.p.

The standard techniques for Noether normalization applied to V/G embedded in K^t via $\pi_{V/G}$ imply the following.

Proposition 3.1 *The problem of constructing an h.s.o.p. for $K[V]^G$ belongs to EXPSPACE unconditionally. (The space is exponential in the bit-length of the succinct specification.) Assuming GRH, it belongs to REXP^{NP}. (Exponentially long oracle queries are allowed.)*

The hierarchy here is exponential and not polynomial because the dimension t of the ambient space K^t containing V/G is exponential in m . The space requirement of this algorithm remains exponential even if we only want to construct an S of $\text{poly}(n)$ size (not necessarily optimal) such that $K[V]^G$ is integral over the subring generated by S .

Proof: Let V/G be the categorical quotient embedded in K^t , $t \leq r^{m^2}$, as in eq.(12). Theorem 2.8 gives a set of exponentially many generators $\{g_1, g_2, \dots\}$ for the ideal of $V/G \subseteq K^t$ whose specifications as straight-line programs can be computed in $\text{poly}(N)$ time, where $N = O(r^{O(m^2)}) =$

$O(2^{\text{poly}(n)})$ is the total bit-length of the straight-line programs. Applying Lemma 2.15 to these generators, we compute a set $S \subseteq K[V]^G$ of invariants of optimal cardinality (equal to $\dim(V/G) \leq n$) such that $K[V]^G$ is integral over the subring generated by S . Then S is an h.s.o.p. This is an EXPSPACE algorithm that works in $\text{poly}(N) = 2^{\text{poly}(n)}$ work-space unconditionally. It is an $REXP^{NP}$ -algorithm assuming GRH. Q.E.D.

The proof of the above result implies that $K[V]^G$ has an h.s.o.p. with exponential total bit-length of specification. It is not known if $K[V]^G$ has an h.s.o.p. with subexponential bit-length of specification. As such, it seems difficult to go any further than Prop 3.1 as far as the construction of an h.s.o.p. is concerned.

Gröbner basis theory yields the following result for the construction of a small separating set.

Proposition 3.2 *A small separating set S of cardinality $\leq 2nm^2$ can be constructed for $K[V]^G$ using space that is exponential in n and time that is double exponential.*

The space and time bounds for this algorithm remain the same even if we only require $|S|$ to be $\text{poly}(n)$. The overall bound does not change assuming GRH. This algorithm can also be modified to yield a separating homogeneous S of degree $\leq 2m^2$ with optimal cardinality. The space and time bounds remain the same.

The article [Ke], specialized to this setting, yields an algorithm for constructing a separating set for $K[V]^G$ that works in time that is exponential in n . But the cardinality of the separating set constructed by this algorithm is also exponential in n .

Proof: This follows by a straightforward constructivization of the proof of Proposition 2.12 using Gröbner basis theory. Let F in that proof be the set of generators of degree $D \leq m^2$ given by Theorem 2.5. Let $X_j[V, G] = \text{spec}(R_j[V, G])$, $1 \leq j \leq D = m^2$, and the set H_j of generators of $R_j[V, G]$ be as in the proof of Proposition 2.12. The set H_j can be used to embed $X_j[V, G]$ in K^{t_j} , $t_j = |H_j|$. Unlike in the proof of Proposition 3.1, where we knew explicit defining equations for V/G , we do not know explicit defining equations for $X_j[V, G]$. Equations for $X_j[V, G]$ can be constructed in space that is exponential in n and time that is double exponential in n using Gröbner basis theory (specifically, using Theorem 1 in [MR2]). The space bound is exponential in n , since t_j is exponential in n . After this, we can construct an h.s.o.p. E_j of $R_j[V, G]$ using Lemma 2.15 in exponential space. The set $S = \{g \mid g(v) - g(w) \in E\}$, $E = \cup_j E_j$, is then a small separating set. Q.E.D.

3.2 Efficient construction of an s.s.o.p.

By Theorem 3.6 below, the double exponential time bound for the construction of a small separating S in Proposition 3.2 can be brought down to polynomial, assuming black-box derandomization hypothesis for STIT, and requiring $|S|$ to be only $O(\text{poly}(n))$ (instead of optimal or $\leq 2nm^2$). We need a few definitions before we can state the result.

Definition 3.3 *We call a set $S \subseteq K[V]^G$ an s.s.o.p. (small system of parameters) for $K[V]^G$ if (1) S contains $\text{poly}(n)$ homogeneous invariants of $\text{poly}(n)$ degree, (2) $K[V]^G$ is integral over its*

subring generated by S , (3) each invariant $s = s(U_1, \dots, U_r)$ in S has a weakly skew straight-line program [MP] over \mathbb{Q} of $\text{poly}(n)$ bit-length.

We say that S is a separating s.s.o.p. if (2) is replaced by the stronger (2)': S is separating (cf. Theorem 2.11).

A (separating) subexponential s.s.o.p. with index $\delta > 0$ is defined by replacing the $\text{poly}(n)$ bounds by $O(2^{O(n^\delta)})$ bounds.

We are not requiring here that the size of S be optimal (equal to $\dim(K[V]^G)$). Given a weakly skew straight-line program for any $s \in S$ and any rational matrices $A_1, \dots, A_r \in M_n(\mathbb{Q})$, the value $s(A_1, \dots, A_r)$ can be computed in time polynomial in n and the total bit-length of the specifications of A_i 's, and even fast in parallel by an AC^0 circuit with an oracle access to DET [MP, Cs].

We say that Noether's normalization lemma for $K[V]^G$ is *derandomized* if there exists an *explicit system of parameters* (e.s.o.p.) for $K[V]^G$ as defined below. We say that it is derandomized in a strong sense if there exists a separating e.s.o.p. for $K[V]^G$.

Definition 3.4 We say that $S \subseteq K[V]^G$ is an e.s.o.p. (explicit system of parameters) if (1) S is an s.s.o.p., and (3) given m and r , the specification of S , consisting of a weakly skew straight-line program as in Definition 3.3 for each $s \in S$, can be computed in $\text{poly}(n)$ time.

We say that S a separating e.s.o.p. if (1) is replaced by the stronger (1)': S is a separating s.s.o.p.

A (separating) subexponential e.s.o.p. with index $\delta > 0$ is defined by replacing the $\text{poly}(n)$ bounds by $O(2^{O(n^\delta)})$ bounds.

The following is the full statement of Theorem 1.1 (b) (1).

Theorem 3.5 Suppose S is a separating e.s.o.p. for $K[V]^G$. Let $\psi_S : V \rightarrow K^k$, $k = |S|$, denote the map

$$\psi_S : v \rightarrow (s_1(v), \dots, s_k(v)),$$

where s_1, \dots, s_k are the elements of S . It can be factored as:

$$V \xrightarrow{\pi_{V/G}} V/G \xrightarrow{\psi'_S} K^k. \quad (13)$$

(1) Given a rational $v \in V$, $\psi_S(v)$ can be computed in time that is polynomial in n and the bit length of the specification of v . If v is not rational then $\psi_S(v)$ can be computed in $\text{poly}(n)$ operations over K .

(2) The image $\psi_S(V) \subseteq K^k$ is a closed subvariety. The map ψ'_S from V/G to $\psi_S(V)$ is one-to-one and onto.

(3) For any $x \in \psi_S(V)$, $\psi_S^{-1}(x)$ contains a unique closed G -orbit in V .

This result holds for any finite dimensional rational representation V of G . The definition of an e.s.o.p. in this general setting is given later (Definition 8.4).

Proof:

- (1) This follows because S is an e.s.o.p.
- (2) By Theorem 2.9 (a), the first map is surjective. The image of the second map is closed because $K[V/G] = K[V]^G$ is integral over the subring generated by S . The map ψ'_S is one-to-one because S is separating.
- (3) This follows from (2) and Theorem 2.9 (b). Q.E.D.

The following is the full statement of Theorem 1.1 (b) (2) and (c).

Theorem 3.6 *Let K be an algebraically closed field of characteristic zero.*

- (1) *Assume that the black-box derandomization hypothesis for STIT (symbolic trace identity testing) over K holds. Then $K[V]^G$ has a separating e.s.o.p.*
- (2) *Suppose STIT for $m \times m$ matrices and r variables has $O(2^{n^\epsilon})$ -time-computable black-box derandomization, $n = rm^2$, for any small constant $\epsilon > 0$. Then $K[V]^G$ has a separating subexponential e.s.o.p. for any exponent $\delta > 0$.*
- (3) *The result in (1) holds, after replacing the $\text{poly}(n)$ bound by $O(n^{\log n})$ bound, if we assume black-box derandomization of PIT for depth four circuits over K instead of black-box derandomization of STIT.*
- (4) *Assuming parallel black-box derandomization of STIT over K (cf. Section 2.1), the problem of constructing a separating s.s.o.p. for $K[V]^G$ belongs to $DET \subseteq NC^2 \subseteq P$.*

Here DET denotes $[C]$ the class of functions that can be computed by AC^0 circuits (of $\text{poly}(n)$ bit-size) with oracle access to the determinant function.

Proof: We only prove (1). The proof of (2) is similar, and so too the proof of (3), since by [AV], black-box derandomization of PIT for depth four circuits over K implies $O(n^{\log n})$ -computable black-box derandomization of low-degree PIT, and hence, STIT. The proof of (4) is similar to that of (1) using the parallel black-box derandomization hypothesis in place of the sequential hypothesis.

For any $l \leq k := m^2$, consider the generic invariant

$$T_l(X, U) = \text{trace}(X_1 \otimes U_1 + \cdots + X_r \otimes U_r)^l, \quad (14)$$

where X_i 's are new $k \times k$ variable matrices, $X = (X_1, \dots, X_r)$, $U = (U_1, \dots, U_r)$, and \otimes denotes the Kronecker product of matrices. Thus each $X_i \otimes U_i$ is an $m' \times m'$ matrix, where $m' = km = m^3$, and for any $A = (A_1, \dots, A_r) \in M_m(K)^r$, $T_l(X, A)$ is a symbolic trace polynomial over an $m' \times m'$ matrix in the rk^2 entries of X_i 's. We have

$$T_l(X, U) = \sum_{\alpha} T_{\alpha}(X) T_{\alpha}(U) = \sum_{[\alpha]} |[\alpha]| T_{[\alpha]}(X) T_{[\alpha]}(U), \quad (15)$$

where $[\alpha] = [\alpha_1 \cdots \alpha_l]$ ranges over the equivalence classes of all words of length l with each $\alpha_j \in [r]$, and $|[\alpha]|$ denotes the cardinality of $[\alpha]$.

Let $U' = (U'_1, \dots, U'_r)$ be another tuple of variable $m \times m$ matrices. For each $l \leq k = m^2$, define the symbolic trace difference

$$\tilde{T}_l(X, U, U') = T_l(X, U) - T_l(X, U'). \quad (16)$$

Each $T_l(X, U, U')$ can be expressed as $\text{trace}(N_l(X, U, U')^{r_l})$ for some symbolic matrix $N_l(X, U, U')$ of size $q = \text{poly}(n)$, $r_l \leq q$, whose entries are (possibly non-homogeneous) linear functions over \mathbb{Q} of the entries of X , U , and U' . This is because STIT is equivalent to PIT for algebraic branching programs (cf. [MP] and Section 2.1), and the difference between two branching programs is again a branching program.

By our black-box derandomization hypothesis for STIT, there exists an explicit ($\text{poly}(n)$ -time computable) hitting set $B = B_{s,q} \subseteq \mathbb{N}^s$ for STIT for $q \times q$ matrices whose entries are linear functions of the $s = rk^2$ variable entries of X_i 's with coefficients in K . Fix such an explicit B . We think of each $b \in B$ as an r -tuple $b = (b_1, \dots, b_r)$ of $k \times k$ integral matrices. By the definition of the hitting set and the argument in the preceding paragraph, for any symbolic trace difference $\tilde{T}_l(X, A, A') = T_l(X, A) - T_l(X, A')$, $l \leq k$, $A = (A_1, \dots, A_r)$, $A' = (A'_1, \dots, A'_r) \in M_m(K)^r$, that is not identically zero as a polynomial in the s variable entries of X_i 's, there exists $b \in B$ such that $\tilde{T}_l(b, A, A') \neq 0$.

Let

$$S = \{T_l(b, U) \mid b \in B, 1 \leq l \leq k\} \subseteq K[V]^G. \quad (17)$$

Suppose $A, A' \in M_m(K)$ are two r -tuples such that for some invariant $h \in K[V]^G$, $h(A) \neq h(A')$. By Theorem 2.5, it follows that some generator $T_{[\alpha]}(U)$ assumes different values at A and A' . By eq. (15) and Corollary 2.7, this implies that $\tilde{T}_l(X, A, A') = T_l(X, A) - T_l(X, A')$ is not identically zero for some $l \leq m^2$. This means there exists $b \in B$ such that $\tilde{T}_l(b, A, A') \neq 0$; i.e., $T_l(b, A) \neq T_l(b, A')$. It follows that S is separating. By Theorem 2.11, it follows that $K[V]^G$ is integral over the subring generated by S . Every element of S is clearly homogeneous of $\text{poly}(n)$ degree. Since the hitting set B is explicit, and matrix powering, Kronecker product, and trace have short and explicit weakly-skew straight-line programs [C, MP], it follows from eq. (14) that the specification of S consisting of a weakly skew straight-line program for its every element can be computed in $\text{poly}(n)$ time. Hence S is a separating e.s.o.p. Q.E.D.

For future reference, we note down the following consequences of the proof.

Lemma 3.7 *Let $X = (X_1, \dots, X_k)$ be new $k \times k$ variable matrices, where $k = m^2$. There exist $\text{poly}(n)$ -time computable weakly skew circuits C_l 's, $l \leq m^2$, over the variable entries of X_i 's and U_i 's such that (1) the polynomial functions $C_l(X, U)$'s computed by C_l 's are of $\text{poly}(n)$ degree, homogeneous in X and U , and can be written as*

$$C_l(X, U) = \sum_{[\alpha]} f_{[\alpha],l}(X) g_{[\alpha],l}(U), \quad (18)$$

where $[\alpha] = [\alpha_1 \dots \alpha_l]$ ranges over the equivalence classes of all words of length l with each $\alpha_j \in [r]$, (2) $f_{[\alpha],l}$'s are linearly independent homogeneous polynomials in the entries of X , and (3) $g_{[\alpha],l}(U)$'s are homogeneous invariants that generate $K[V]^G$.

This result, a key ingredient in the proof of Theorem 3.6, says that the variety V/G here is strongly explicit as per the general notion of explicit varieties that we shall formulate later (Definition 10.2) taking V/G as a basic prototype of such varieties.

Proof: Let C_l be a weakly skew circuit [MP] computing $T_l(X, U)$, cf. eq. (14), so that $C_l(X, U) = T_l(X, U)$, $f_{[\alpha],l}(X) = |[\alpha]|T_{[\alpha]}(X)$, and $g_{[\alpha],l}(U) = T_{[\alpha]}(U)$; cf. eq. (15). Then $f_{[\alpha],l}(X)$'s are linearly independent by Corollary 2.7, and $g_{[\alpha],l}(U)$'s generate $K[V]^G$ by Theorem 2.5. Q.E.D.

Theorem 3.8 *The problem of deciding if the closures of the G -orbits of two rational points in V intersect belongs to co-RNC.*

Proof: By Theorem 2.9 (d) and the proof of Theorem 3.6, the closures of the G -orbits of $A, A' \in V$ intersect iff the symbolic trace difference $\tilde{T}_l(X, A, A') = T_l(X, A) - T_l(X, A')$ is identically zero for some $l \leq k = m^2$. For rational A and A' , this can be tested by a co-RNC algorithm [IM]: just substitute large enough random integer values for the entries of X and test if all the differences vanish. Q.E.D.

3.3 Existence of an s.s.o.p.

The following is the first half of Theorem 1.1 (a).

Theorem 3.9 *The ring $K[V]^G$ has a separating s.s.o.p.*

Proof: Observe that STIT is a special case of the PIT treated in Theorem 2.1. In the proof of Theorem 3.6, use the new hitting set B provided by Theorem 2.1, letting s, r , and d there be $\text{poly}(n)$ (large enough), in place of the old hitting set B provided by the black-box derandomization hypothesis for STIT. Let S be as in (17) with this new B .

Since we do not have any efficient algorithm for constructing B anymore, we do not have any efficient algorithm for constructing S either. Hence the proof of Theorem 3.6 will only show now that S is a separating s.s.o.p. instead of a separating e.s.o.p. Q.E.D.

Let us note down the following consequence of the proof.

Lemma 3.10 *Let B be any hitting set provided by Theorem 2.1 (with s, r , and d as in the proof of Theorem 3.9). Let S be defined as in (17) with this B . Then S is a separating s.s.o.p.*

The following result proves the second half of Theorem 1.1 (a) and Theorem 1.1 (d) (1).

Theorem 3.11 *Let V and G be as in Theorem 3.6. A separating s.s.o.p. can be constructed in $\text{poly}(n)$ work-space. This means the problem of constructing a separating s.s.o.p. belongs to PSPACE unconditionally. The problem belongs to $\Sigma_3 \subseteq PH$ assuming GRH.*

For the proof we need the following lemma.

Lemma 3.12 *Let B be any potential hitting set in the setting of Theorem 2.1 (with s, r , and d as used in the proof of Theorem 3.9). Let $S(B)$ be as in (17) with this B . Then whether $S(B)$ is a separating s.s.o.p. can be verified in $\text{poly}(n)$ space. Assuming GRH, the problem of verification belongs to Π^2 .*

Proof: Let B be as given. Then

$$S(B) = \{T_l(b, U) | b \in B, 1 \leq l \leq k\} \subseteq K[V]^G.$$

Each element of $S(B)$ is homogeneous of $\text{poly}(n)$ degree, has a weakly skew straight-line program of $\text{poly}(n)$ bit-length, and the cardinality of $S(B)$ is $\text{poly}(n)$. So it suffices to check that $S(B)$ is separating.

Let $V^2 = V \times V$. Let $U = (U'_1, \dots, U'_r)$ be another tuple of $m \times m$ variable matrices whose entries are the coordinates of the second copy of V . Let $Z(B) \subseteq V^2$ be the zero set of

$$\tilde{S}(B) = \{T_l(b, U) - T_l(b, U') | b \in B, 1 \leq l \leq k\} \subseteq K[V^2].$$

Let $F = \{T_{[\alpha]}(U)\}$ be the set of generators for $K[V]^G$ as in eq. (11). Then $S(B)$ is separating iff

(*) For every equivalence class $[\alpha]$ of words in $1, \dots, r$ of length $\leq m^2$: (**) $T_{[\alpha]}(U) - T_{[\alpha]}(U')$ vanishes on $Z(B)$.

Using the PSPACE-algorithm for Hilbert's Nullstellansatz [Ko, Ko1], the test (**), for any given α , can be carried out in $\text{poly}(n)$ space. Using the PH-algorithm for Hilbert's Nullstellansatz in Theorem 2.14, it can be carried out by a Π_2 -algorithm assuming GRH. Thus the test (*) can be carried out by a PSPACE algorithm unconditionally, and by a Π_2 -algorithm assuming GRH. Q.E.D.

Proof of Theorem 3.11

Using Lemma 3.12, the problem of checking if there exists $B \subseteq [u]^r$ (with s, r, d , and $u = 2s(d+1)^2$ as in the proof of Theorem 3.9) such that $S(B)$ is a separating s.s.o.p. belongs to PSPACE unconditionally, and to Σ_3 assuming GRH. The algorithm for this check is bound to succeed by Theorem 3.9. Q.E.D.

3.4 Instability flag and decomposition

In this section we prove Theorem 1.1 (e) and (f). Let V, G , and K be as in Theorem 3.6.

The following is a full statement of Theorem 1.1 (e) and (f).

Theorem 3.13 *Let K be an algebraically closed field. Let V and G be as in Theorem 1.1.*

(a) *Given a rational $v \in V$, the instability flag $[Km] \phi(v)$ of an optimal one-parameter subgroup that drives v to a point in the unique closed orbit that lies in the closure of the G -orbit of v can be computed in time that is polynomial in n and the bit-length of the specification of v . If v is not rational, this flag can be computed in $\text{poly}(n)$ operations over K .*

(b) *Given a rational semi-simple $v \in V$, the decomposition of v into simple tuples can also be computed in time that is polynomial in n and the bit-length of the specification of v . If v is not rational, this decomposition can be computed in $\text{poly}(n)$ operations over K .*

The instability flag is specified by the coordinates of the vectors constituting the flag. All coordinates are specified by straight-line programs over some variables $\lambda_1, \dots, \lambda_k$, $k = \text{poly}(n)$,

where each λ_i stands for an algebraic number with the minimal polynomial over \mathbb{Q} of $\text{poly}(n)$ degree. Each λ_i is specified by its minimal polynomial and a complex approximation to it of a high enough precision to distinguish it from other λ_j 's with the same minimal polynomial. We cannot specify the coordinates of the vectors as elements of a common extension field containing λ_i 's, since the degree of such a field can be exponential.

Proof: The point v corresponds to a tuple (A_1, \dots, A_r) of rational $m \times m$ matrices. Consider the representation

$$\phi_v : \mathbb{C}\langle U_1, \dots, U_r \rangle \rightarrow M_m(K),$$

given by $U_i \rightarrow A_i$. Two representations ϕ_v and ϕ_w are isomorphic iff v and w belong to the same G -orbit. By Artin [Ar], closed orbits correspond to semi-simple representations and the representation corresponding to the unique closed orbit in the closure of the G -orbit of v is the semi-simplification of ϕ_v , i.e., the direct sum of its Jordan-Holder components. The flag $\phi(v)$ corresponding to this Jordan-Holder series in the specification we are considering can be computed in polynomial time using the standard techniques for decomposing finite dimensional algebras over algebraically closed fields; cf. [FR] and Theorem 3.5. in [Eb]. In v is semi-simple, the Jordan-Holder components yield its decomposition into simple tuples. The standard techniques yield this decomposition as well in polynomial time. Q.E.D.

3.5 Explicit coarse classification assuming black-box derandomization for any algebra

Next we note down the consequences of Theorems 3.5, 3.6, and 3.13 for arbitrary finite-dimensional algebras.

Let R be a finite dimensional associative algebra over K , specified by its basis f_1, \dots, f_r and the multiplicative structure constants. Let $\rho : R \rightarrow M_m(K)$ be its m -dimensional representation. Let $n = mr^2$. Two representations are isomorphic iff they lie in the same G -orbit, $G = SL_m(K)$. The representations with closed G -orbits are called stable or semi-simple. Each m -dimensional representation ρ of R can be identified with the r -tuple $(A_1, \dots, A_r) \in V = M_m(K)^r$ of $m \times m$ matrices, $A_i = \rho(f_i)$. The set $W_m = W_m(R)$ of m -dimensional representations of R is a closed G -subvariety of V .

The following result says that black-box derandomization of STIT implies explicit coarse classification (parametrization and decomposition) of the finite dimensional semi-simple representations of A .

Theorem 3.14 *Suppose black-box derandomization hypothesis for STIT holds. Then:*

- (a) *There exists a separating e.s.o.p. $S = S_m$ for $K[V]^G$ for any m . This yields a one-to-one and onto explicit (polynomial time computable) map ψ_S from the closed G -orbits in W_m to the points of $\psi_S(W_m) \subseteq K^k$, $k = |S| = \text{poly}(n)$. Furthermore, $\psi_S(W_m)$ is a closed subvariety of K^k .*
- (b) *Each $\rho \in W_m$ can be associated an explicit (polynomial time computable) instability flag $\phi(\rho)$ of an optimal one-parameter subgroup driving ρ to the unique closed G -orbit in the closure of the G -orbit of ρ .*
- (c) *Given a rational semi-simple $\rho \in W_m$, the decomposition of ρ into simple representations can also be computed in time that is polynomial in n and the bit-length of the specification of ρ .*

If ρ is not rational, this decomposition can be computed in $\text{poly}(n)$ operations over K .

This result follows from Theorems 3.6, 3.13, and the following result.

Proposition 3.15 *Suppose S is a separating e.s.o.p. of $K[V]^G$. Then:*

- (a) *Given any rational m -dimensional representation ρ of R , $\psi_S(\rho)$ can be computed in time polynomial in n and the bit-length of the specification of ρ . If ρ is not rational, then $\psi_S(\rho)$ can be computed in $\text{poly}(n)$ arithmetic operations over K .*
- (b) *The image $W'_m = \psi_S(W_m)$ is a closed subvariety of K^k , $k = |S| = \text{poly}(n)$.*
- (c) *For any $x \in W'_m$, $\psi_S^{-1}(x)$ contains a unique closed G -orbit in W_m .*

Proof:

- (a) This follows since S is an e.s.o.p.
- (b) By Theorem 2.9 (c), $Y = \pi_{V/G}(W_m)$ is a closed subvariety of V/G . The image $\psi'_S(Y)$, with ψ'_S as in eq. (13), is closed since $K[V/G] = K[V]^G$ is integral over the subring generated by S and hence ψ'_S is a finite morphism.
- (c) This follows from Theorem 3.5 (3). Q.E.D.

4 Generalization to quivers

We now generalize Theorems 3.6 and 3.8 to arbitrary quivers.

Let Q be a quiver [BP, DW], i.e., a four-tuple (Q_0, Q_1, t, h) , where $Q_0 = \{1, \dots, l\}$ is a set of vertices, Q_1 is a finite set of arrows between these vertices, and the two maps $t, h : Q_1 \rightarrow Q_0$ assign to each arrow $\phi \in Q_1$ its tail $t(\phi)$ and head $h(\phi)$. Loops and multiple arrows are allowed. A representation W of the quiver Q over a field K is a family $\{W(i) : i \in Q_0\}$ of finite dimensional vector spaces over K together with a family of linear maps $W(\phi) : W(t(\phi)) \rightarrow W(h(\phi))$, $\phi \in Q_1$. The l -tuple of integers $\dim(W) = \{\dim(W(i)) \mid i \in Q_0\}$ is called the dimension vector of W . A morphism between two representations $f : W_1 \rightarrow W_2$ is a family of linear morphisms $\{f(i) : W_1(i) \rightarrow W_2(i) \mid i \in Q_0\}$ such that, for all $\phi \in Q_1$, $W_2(\phi) \circ f(t(\phi)) = f(h(\phi)) \circ W_1(\phi)$. For a fixed dimension vector $m = (m(1), \dots, m(l)) \in \mathbb{N}^l$, the representation space $V = V(Q, m)$ of the quiver Q is the set of all representations W of Q such that $W(i) = K^{m(i)}$ for all $i \in Q_0$. Clearly,

$$V = V(Q, m) = \bigoplus_{\phi \in Q_1} \text{Hom}_K(K^{m(t(\phi))}, K^{m(h(\phi))}) = \bigoplus_{\phi \in Q_1} M_\phi(K), \quad (19)$$

where $M_\phi(K)$ denotes the space of $m(h(\phi)) \times m(t(\phi))$ matrices with entries in K . There is a canonical action of

$$G = \prod_{i=1}^l GL_{m(i)}(K)$$

on V defined by

$$(g \cdot W)(\phi) = g(h(\phi))W(\phi)g(t(\phi))^{-1}$$

for any $g = (g(1), \dots, g(l)) \in G$ and $W \in V(Q, m)$. The G -orbits in V are precisely the isomorphism classes of representations of Q with the dimension vector m .

Assume that K is of characteristic zero. Let $U = (\dots, U_\phi, \dots)$ be the tuple of variable matrices, where U_ϕ is a $m(h(\phi)) \times m(t(\phi))$ variable matrix. Then the coordinate ring $K[V]$ of V can be identified with the ring $K[U]$ over the variable entries of U_ϕ 's. Let $K[V]^G \subseteq K[V]$ be the subring of G -invariants. When Q consists of a single vertex with r self-loops and dimension m , $K[V]^G$ coincides with the invariant ring in Theorem 3.6. If Q has no directed cycles then $K[V]^G = K$. So we are mainly interested in the case when Q has directed cycles.

The following result generalizes Theorem 3.6 to arbitrary quivers.

Theorem 4.1 *Let K be an algebraically closed field of characteristic zero, and $n = |Q_0| + |Q_1| + |m|$, where $|m| = \sum_{i \in Q_0} m(i)$. Then:*

- (a) *The analogue of Theorem 3.6 holds for V and G as above.*
- (b) *Analogues of Theorems 3.8, 3.9, and 3.11 also hold.*

For a proof, we need the following generalization of Theorem 2.5.

Theorem 4.2 (Le Bruyn-Procesi) *[BP] The ring $K[V]^G$ is generated by traces of the form $\text{trace}(U_{\phi_1} \cdots U_{\phi_l})$, $l \leq |m|^2$, where (ϕ_1, \dots, ϕ_l) , $\phi_j \in Q_1$, denotes a directed cycle in the quiver Q .*

Proof of Theorem 4.1:

- (a) The proof is very similar to that of Theorem 3.6. So we only indicate the differences.

For any cycle $\alpha = (\phi_1, \dots, \phi_l)$, $\phi_j \in Q_1$, $l \leq |m|^2$, in the quiver Q , let $T_\alpha(U) = \text{trace}(U_{\phi_1} \cdots U_{\phi_l})$. We call two cycles equivalent if one is obtained by a cyclic rotation from the other. Let $[\alpha]$ denote the equivalence class of α , and let $T_{[\alpha]}(U) = T_\alpha(U)$. The generalization of F in eq.(11) is now

$$F = \{T_{[\alpha]}(U)\}, \quad (20)$$

where $[\alpha]$ ranges over the equivalence classes of cycles of length $\leq |m|^2$. By Theorem 4.2, F generates $K[V]^G$. Let $t = |F|$. The map $\pi_{V/G} : V \rightarrow K^t$ in (12) now generalizes to

$$\pi_{V/G} : A = (\dots, A_\phi, \dots) \rightarrow (\dots, T_{[\alpha]}(A), \dots). \quad (21)$$

For any $l \leq k := |m|^2$, the generic invariant in (14) now generalizes to

$$T_l(X, U) = \text{trace}(M(X, U)^l), \quad (22)$$

where $X = (\dots, X_\phi, \dots)$ is a tuple of new $k \times k$ variable matrices X_ϕ 's, $M(X, U)$ denotes the $|Q_0| \times |Q_0|$ block matrix whose (i, j) -th block, $i, j \in Q_0$, is the matrix $\sum_\phi X_\phi \otimes U_\phi$, where ϕ ranges over all arrows in Q_1 with tail i and head j , and \otimes denotes the Kronecker product. For any $A = (\dots, A_\phi, \dots) \in V$, $T_l(X, A)$ is a symbolic trace polynomial over an $m' \times m'$ matrix, $m' = \dim(M(X, U)) = \text{poly}(n)$, in the $s = \sum_\phi \dim(X_\phi)^2 = \text{poly}(n)$ variable entries of X_ϕ 's.

The rest of the proof proceeds as that of Theorem 3.6 with $T_l(X, U)$ in (22) in place of $T_l(X, U)$ in (14).

(b) The proof of (a) can be extended just as the proof of Theorem 3.6 is extended to prove Theorems 3.8, 3.9, and 3.11. Q.E.D.

Conjecture 4.3 *Analogue of Theorem 3.13 (a) also holds for V and G as in Theorem 4.1.*

The proof of Theorem 3.13 can not be extended as it is since a characterization of semi-simple representations of a quiver analogous to Artin's characterization [Ar] of semi-simple tuples is not known.

5 Implications of lower bounds for EXP in characteristic zero

In this section we prove Theorem 1.2 (a), (b), and a half of (f). The following is its full statement.

Theorem 5.1 *Let K be any algebraically closed field of characteristic zero. Assume that V and G are as in Theorem 3.6, the case in Theorem 4.1 being similar.*

(1) *Suppose there exists an exponential-time-computable multilinear polynomial $f(x_1, \dots, x_r)$ with integral coefficients of $\text{poly}(r)$ bit-length such that f can not be evaluated by an arithmetic circuit over K of size $O(2^{r^a})$ and depth $O(r^a)$, for some constant $a > 0$. Then $K[V]^G$ has a separating quasi-e.s.o.p.*

(2) *Suppose f as in (1) can not be evaluated by an arithmetic circuit over K of size $O(r^a)$ and depth $O(\log^2 r)$, for any constant $a > 0$. Then $K[V]^G$ has a separating subexponential e.s.o.p. for any index $\delta > 0$.*

(3) *The implication in (1) also holds if there can not be evaluated by an arithmetic circuit over K of size $O(2^{r^{a'}})$ and constant depth, for some constant $a' > 0$.*

(4) *The implication in (2) also holds if f there cannot be computed by an arithmetic circuit over K of constant depth and size $O(n^{O(\sqrt{n} \log n)})$.*

(5) *The implications in (1)-(4) hold if f is instead exponential-time-computable with access to the NP-oracle, giving the algorithm for constructing S an access to the NP-oracle in this case.*

For a proof we need the following result.

Theorem 5.2 (Agrawal, Vinay; Koiran) *Let K be a field of characteristic zero.*

(a) [Agrawal, Vinay], cf. [AV]: *A multilinear polynomial in r variables that can be computed by an arithmetic circuit over K of $2^{o(r)}$ size can also be computed by a depth four arithmetic circuit over K of $2^{o(r)}$ size.*

(b) [Koiran], cf. [Ko2]: *A multilinear polynomial in r variables that can be computed by an arithmetic circuit over K of $\text{poly}(r)$ size can also be computed by a depth four arithmetic circuit over K of $O(r^{O(\sqrt{r} \log r)})$ size.*

Proof of Theorem 5.1:

(1) The lower bound assumption implies that f can not be evaluated by an arithmetic circuit over K of size $O(2^{r^{a'}})$, $a' = a/2$, without any depth restriction. To see this, suppose to the contrary, that f can be evaluated by an arithmetic circuit C' over K of size $s = O(2^{r^{a'}})$. Applying the parallelization technique in [VSBR] to C' , it follows that f can be evaluated by an arithmetic circuit C over K of size $\text{poly}(s) = O(2^{r^a})$ and depth $O((\log s + \log(\deg(f))) \log s) = O(r^a)$; a contradiction.

It now follows from Theorem 2.2 (a) that PIT for small degree circuits of size $\leq s$ has $O(2^{\text{polylog}(s)})$ -time computable black-box derandomization. In particular, STIT for $m \times m$ symbolic matrices and r variables has $O(2^{\text{polylog}(s)})$ -time computable black-box derandomization, $s = rm^2$. Now we can apply Theorem 3.6 (1), or rather its proof, with the following modification. In Theorem 3.6 (1) it is assumed that STIT has polynomial time computable black-box derandomization, rather than quasi-polynomial time computable black-box derandomization as here. But it can be checked that the proof of Theorem 3.6 (1) goes through replacing polynomial bounds by quasi-polynomial bounds.

(2) This follows similarly from Theorem 2.2 (b) and Theorem 3.6 (2).

(3) By Theorem 5.2 (a), the assumption in (3) implies the assumption in (1). So (3) follows from (1).

(4) By Theorem 5.2 (b), the assumption in (4) implies the assumption in (2). So (4) follows from (2).

(5) The proof of (1)-(4) relativizes. Q.E.D.

The following result is a consequence of the proof of Theorem 5.1 (1) letting f in Theorem 2.2 (a) be the permanent function.

Theorem 5.3 *Let V, G , and K be as in Theorem 5.1. Suppose $\text{perm}(X)$, X a variable $m \times m$ matrix, cannot be computed by arithmetic circuits over K of $O(2^{m^a})$ size and $O(m^a)$ depth for some constant $a > 0$. Let B be the hitting set for STIT for symbolic matrices of (large enough) $\text{poly}(n)$ size and $\text{poly}(n)$ variables constructed using the arithmetic NW-generator $[NW, KI]$ based on the permanent as a hard function. Then $\{T_l(b, U) \mid b \in B, l \leq k = m^2\}$, cf. eq. (17), is a separating quasi-e.s.o.p. for $K[V]^G$.*

6 Implications of Boolean lower bounds for EXP

In this section we prove Theorem 1.2 (c), (d), (e), and the second half of (f). The following is its full statement.

Theorem 6.1 *Assume that the Generalized Riemann Hypothesis (GRH) holds. Let K be any algebraically closed field of characteristic zero.*

Assume that V and G are as in Theorem 3.6, the case in Theorem 4.1 being similar.

(1) Suppose $\text{EXP} \not\subseteq \text{i.o.nonuniform-NC}[n^\epsilon, 2^{n^\epsilon}]$, for some constant $\epsilon > 0$, or alternatively, that $\text{EXP} \not\subseteq \text{i.o.TC}^0[2^{n/a}]$, for some constant $a > 1$. Then $K[V]^G$ has a separating quasi-e.s.o.p.

The prefix i.o. can be removed from the assumption and added to the conclusion. This means the conclusion then holds for infinitely many n .

(2) Suppose $EXP \not\subseteq \text{i.o.nonuniform-NC}^3$, or alternatively, that $EXP \not\subseteq \text{i.o.TC}^0[n^{O(\sqrt{n} \log n)}]$. Then $K[V]^G$ has a separating subexponential e.s.o.p. for any index $\delta > 0$. The prefix i.o. can be removed from the assumption and added to the conclusion.

(3) Analogues of (1) and (2) also hold if we replace EXP by EXP^{NP} and give the algorithms for computing S an access to the NP-oracle.

(4) If $EXP \not\subseteq MA$ then there is a subexponential time algorithm for constructing a separating subexponential-s.s.o.p. for $K[V]^G$, for any exponent $\delta > 0$, that is correct for infinitely many n . If $EXP^{NP} \not\subseteq \Sigma_2^P \cap \Pi_2^P$, or $NEXP \not\subseteq MA$, then there is a subexponential time algorithm for constructing a separating subexponential-s.s.o.p. for $K[V]^G$, for any exponent $\delta > 0$, that is correct for infinitely many n , giving the algorithm for constructing S an access to the NP-oracle.

For a proof, we need the following result based on Bürgisser (Theorem 2.13).

Lemma 6.2 *Let f be an n -variate polynomial of $\text{poly}(n)$ degree with non-negative integral coefficients of $\text{poly}(n)$ bit-length. Let K be any field of characteristic zero.*

(a) *Suppose f can be computed by a nonuniform arithmetic circuit C over K of size $s = O(2^{\text{poly}(n)})$. Then, assuming GRH, there exists a nonuniform Boolean circuit C_2 of $\text{poly}(s)$ size and $O(\log^2(s) \log(n))$ depth such that $C_2(x) = f(x) \pmod{2}$, for all $x \in \{0, 1\}^n$.*

(b) *Suppose f can be computed by a nonuniform constant-depth (unbounded fan-in) arithmetic circuit C over K of size $s = O(2^{\text{poly}(n)})$. Then, assuming GRH, there exists a Boolean constant-depth threshold circuit C_2 of $\text{poly}(s)$ size such that $C_2(x) = f(x) \pmod{2}$, for all $x \in \{0, 1\}^n$.*

Proof:

(a) The proof is very similar to that of Theorem 4.5 in [Bu]. Suppose f can be computed by a nonuniform arithmetic circuit C over K of size $s = O(2^{\text{poly}(n)})$. Without loss of generality, we can assume that C has $O(\log^2(s))$ depth. Otherwise, using the parallelization technique in [VSBR], C can be replaced with an equivalent circuit of size $\text{poly}(s, n) = \text{poly}(s)$ and depth $O((\log s)(\log s + \log(\deg(f)))) = O(\log^2(s)) = O(\text{poly}(n))$. Let \tilde{C} be the circuit obtained from C by replacing each constant $c_i \in K$ in C by a new variable y_i . Let $y = (y_1, \dots, y_q)$ be the tuple of the newly added variables, and $c = (c_1, \dots, c_q)$ the tuple of the replaced constants. Then $\tilde{C}(x, y)$ is an integral function with the degree as well as the logarithm of the weight bounded by $2^{\text{poly}(n)}$. Clearly, $\tilde{C}(x, c) = C(x) = f(x)$. Hence, the integral system of polynomial equations over the variables y

$$(S_n): \quad \tilde{C}(x, y) - f(x) = 0 \quad \text{for all } x \in \{0, \dots, 1\}^n$$

has a solution $y = c \in K^q$. By Hilbert's Nullstellensatz, (S_n) also has a solution over the algebraic closure of \mathbb{Q} and hence over \mathbb{C} . Since the degree of $f(x)$ is $\text{poly}(n)$ and the coefficients of $f(x)$ are non-negative integers with $\text{poly}(n)$ bit-length, $0 \leq f(x) \leq 2^{n^a}$, for all $x \in \{0, \dots, 1\}^n$, for some constant $a > 0$. Assuming GRH, by Theorem 2.13, $\pi_{S_n}(2^{n^b}) > 2^{n^a}$ for some constant $b > 0$, $n \rightarrow \infty$. Hence, there is some prime p_n satisfying $n^a < \log_2(p_n) \leq n^b$ such that (S_n) is solvable over F_{p_n} . Let $\bar{y}_n \in F_{p_n}^q$ denote such a solution. Let C_{p_n} denote the circuit over F_{p_n} obtained by substituting $y = \bar{y}_n$ in \tilde{C} . Its size is $\text{poly}(s)$ and depth is $O(\log^2(s))$. For any

$x \in \{0, 1\}^n$, $0 \leq f(x) \leq 2^{n^a} < p_n$, and hence, $f(x) = C_{p_n}(x)$ by (S_n) , and $f(x) \pmod{2}$ is the last bit in the binary representation of $C_{p_n}(x)$.

The addition and multiplication in F_{p_n} can be performed by Boolean circuits of size $(\log p_n)^{O(1)} = \text{poly}(n)$ and depth $O(\log \log p_n) = O(\log(n))$. Hence, using C_{p_n} , we can construct a Boolean circuit C'_{p_n} of $\text{poly}(s)$ size and $O(\log^2(s) \log(n))$ depth such that the output of $C'_{p_n}(x)$, for all $x \in \{0, 1\}^n$, is the binary representation of $f(x)$. The last bit of this output is $f(x) \pmod{2}$. Hence, using C'_{p_n} , we get the required Boolean circuit C_2 that just outputs the last bit of $C'_{p_n}(x)$.

(b) The proof is similar to that of (a). The circuits C, \tilde{C} , and C_{p_n} now have constant depth. Since iterated addition, iterated multiplication and division are in TC^0 [HAB], the unbounded fan-in addition and multiplication gates over F_{p_n} in C_{p_n} can be simulated by Boolean constant depth threshold circuits of $O(\text{poly}(s))$ size. This yields a Boolean constant-depth threshold circuit C_2 of $O(\text{poly}(s))$ size. Q.E.D.

Proof of Theorem 6.1:

(1) If we take the first assumption, then there exists a Boolean function $h(x) = h(x_1, \dots, x_n) \in EXP$ that is not in $\text{i.o.nonuniform-NC}[n^\epsilon, 2^{n^\epsilon}]$ for some constant $\epsilon > 0$. This means $h(x)$ does not have a non-uniform circuit of depth n^ϵ and size 2^{n^ϵ} for all large enough n . If we take the second alternative assumption instead, then h is not in $\text{i.o.TC}^0[2^{n/a}]$ for some constant $a > 1$. This means h does not have a TC^0 circuit of size $2^{n/a}$ for all large enough n .

Since x_i 's here are Boolean variables, we can write h as

$$h(x) = \sum_{\mu} a_{\mu} \mu, \quad (23)$$

where μ ranges over all monomials in x_i 's, with degree in each x_i either 0 or 1, and $a_{\mu} \in \{0, 1\}$. Define multilinear $f(x) = f(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$ by

$$f(x) = \sum_{\mu} a_{\mu} \mu, \quad (24)$$

where μ and a_{μ} are exactly as in eq.(23). We only think of the variables x_i 's as integer variables now, instead of Boolean variables. Clearly, $f = h, \text{ mod } 2$. Since $h \in EXP$, the expanded form (23) of $h(x)$ can be computed in exponential time. After this, given any input $b \in \mathbb{Z}^n$, $f(b) \in \mathbb{Z}$ can be computed in time that is exponential in n and the bit-length of b .

Claim 6.3 *Assuming GRH, the integral polynomial f cannot be evaluated by an arithmetic circuit of size $O(2^{n^{\epsilon'}})$, $\epsilon' > 0$ a small enough constant, $n \rightarrow \infty$, over any field K of characteristic zero, if we take the first assumption. It cannot be evaluated by an arithmetic circuit over K of size $2^{o(n)}$, if we take the second assumption.*

Proof of the claim:

Take the first assumption. Suppose to the contrary that f can be evaluated by an arithmetic circuit over K of size $O(2^{n^{\epsilon'}})$. By Lemma 6.2 (a), $h(x)$ can be computed by a Boolean circuit of size $O(2^{an^{\epsilon'}})$ and depth $O(n^{b\epsilon'})$, for some constants $a, b > 0$. Choosing ϵ' small enough, this contradicts our assumption that h is not in $\text{i.o.nonuniform-NC}[n^\epsilon, 2^{n^\epsilon}]$.

Now take the second assumption. Suppose to the contrary that f can be evaluated by an arithmetic circuit over K of size $O(2^{o(n)})$. By Theorem 5.2 (a), f can be evaluated by a depth four circuit over K of size $O(2^{o(n)})$. By Lemma 6.2 (b), $h(x)$ can be computed by a constant-depth Boolean threshold circuit of size $2^{o(n)}$. This contradicts our assumption that h is not in $i.o.TC^0[2^{n/a}]$.

This proves the claim.

Now (1) follows from this claim and Theorem 5.1 (1).

If we remove the prefix i.o. from the assumption, then the proof goes through as long as we add the prefix i.o. to the conclusion. The changes are straightforward and we omit the details.

(2) For the first assumption, this follows similarly from Theorem 5.1 (2) and Lemma 6.2 (a). For the second assumption, this follows similarly from Theorem 5.1 (4) and Lemma 6.2 (b).

(3) This follows by relativizing the proofs of (1) and (2).

(4) By Babai, Fortnow, Nisan, and Wigderson [BFNW], $EXP \not\subseteq MA$ implies $EXP \not\subseteq P/poly$ and hence $EXP \not\subseteq \text{nonuniform-NC}^3$. Hence the first statement follows from (2). By Buhrman and Homer [BuH], $EXP^{NP} \not\subseteq \Sigma_2^P \cap \Pi_2^P$ implies $EXP^{NP} \not\subseteq P/poly$, and by Impagliazzo, Kabanets, and Wigderson [IKW], $NEXP \subseteq P/poly$ iff $NEXP = MA$. Hence the second statement follows from (3). Q.E.D.

7 The implications of GRH alone

We now put all the ingredients in the preceding sections together to prove Theorem 1.1 (d) (2) and (3). The following is the full statement of Theorem 1.1 (d) (2).

Theorem 7.1 *Suppose GRH holds.*

Let V and G be as in Theorem 3.6, the case in Theorem 4.1 being similar. Then the problem of derandomizing Noether's normalization lemma for $K[V]^G$ in a strong form belongs to $\Delta_3^{\text{quasi-}P}$. This means $K[V]^G$ has a separating quasi-e.s.o.p., giving the algorithm for constructing S an access to the NP^{NP} -oracle.

For the proof we need the following last ingredient.

Theorem 7.2 (Kannan, Miltersen, Vinodchandran, Watanabe) *(cf. [Kn, MVW]) There is a language in $\Delta_3^e = ETIME^{NP^{NP}}$ that does not have circuits of size $2^{n/a}$ for all n , for some constant a .*

Proof of Theorem 7.1:

Relativizing the proof of Theorem 6.1 (1) with respect to the NP^{NP} -oracle, it follows that $K[V]^G$ has a quasi-e.s.o.p., giving the algorithm for constructing S an access to the NP^{NP} -oracle, assuming that (a) GRH holds, and (b) some function in $EXP^{NP^{NP}}$ does not have subexponential size circuits for all n . The assumption (b) now holds unconditionally by Theorem 7.2. Q.E.D.

This proof also yields:

Theorem 7.3 *Assume that GRH holds. Let K be an algebraically closed field of characteristic zero. Then the black-box derandomization problem for PIT for small degree circuits over K belongs to $\Delta_3^{\text{quasi-P}}$. This means the PIT for small degree circuits over K has $O(2^{\text{polylog}(s)})$ -time computable derandomization with access to the NP^{NP} -oracle.*

The problem also belongs PSPACE unconditionally and to Σ^3 assuming GRH; cf. Theorem 10.9.

The following is the full statement of Theorem 1.1 (d) (3).

Theorem 7.4 *Let V and G be as in Theorem 3.6, the case in Theorem 4.1 being similar. Suppose GRH holds. Then the problem of derandomizing Noether's normalization lemma for $K[V]^G$ in a strong form also belongs to $i.o.MA_{\text{quasi-half-EXP}}$. This means there is an $MA_{\text{quasi-half-EXP}}$ -algorithm for constructing a separating quasi-half-exponential s.s.o.p. as defined below that works correctly for infinitely many n .*

Here we think of MA as a class of functions rather than decision problems. A quasi-half-exponential function is a function of the form $e((e_{-1/2}(n^a))^b)$, where a and b are constants, and $e_\alpha(n)$, for a rational α , is the α 'th iterate of the exponential function $e(n)$ as defined in [MVW]. A function of the form $e_{1/2}(n^a)$ is called a half-exponential function. A quasi-half-exponential s.s.o.p. is defined like a quasi s.s.o.p. using quasi-half-exponentials instead of quasi-polynomials.

Proof: The result is proved just like Theorem 7.1, using Theorem 7.5 below in place of Theorem 7.2 and Theorem 7.6 below in place of Theorem 2.2 (a). Q.E.D.

Theorem 7.5 (Miltersen, Vinodchandran, Watanabe) *(cf. [MVW]) There is a language in MA_{EXP} that does not have circuits of half-exponential size for infinitely many n .*

Also see a related bound in Santhanam [Sa]. If this result can be strengthened to show that MA_{EXP} does not have sub-exponential size circuits for all n , then it would follow from the proof of Theorem 7.4 that the problem of derandomizing Noether's Normalization Lemma for $K[V]^G$ in a strong form is in MA assuming GRH.

Theorem 7.6 *If f as in Theorem 2.2 cannot be evaluated by an arithmetic circuit over K of half-exponential size, then PIT for small degree circuits over K has quasi-half-exponential-time-computable black-box derandomization.*

The proof is similar to that of Theorem 2.2.

The proof of Theorem 7.4 also yields the following result.

Theorem 7.7 *Suppose GRH holds. Let K be an algebraically closed field of characteristic zero. Then the black-box derandomization problem for PIT for small degree circuits over K also belongs to $i.o.MA_{\text{quasi-half-EXP}}$.*

8 The general ring of invariants

In this section we prove Theorems 1.3 and 1.4.

Let K be an algebraically closed field of characteristic zero. Let V be a polynomial representation of $G = SL_m(K)$ of degree $\leq d$ and dimension n . Since G is reductive [Fu], V can be decomposed as a direct sum of irreducibles:

$$V = \oplus_{\lambda} m(\lambda) V_{\lambda}(G). \quad (25)$$

Here $\lambda : \lambda_1 \geq \dots \geq \lambda_r > 0$, $r < m$, is a partition, i.e., a non-increasing sequence of positive integers, $V_{\lambda}(G)$ is the irreducible Weyl module [Fu] of G labelled by λ , and $m(\lambda)$ is its multiplicity. The degree d is the maximum of $|\lambda| = \sum_i \lambda_i$ for the λ 's that occur in this decomposition with nonzero multiplicity. It is easy to see that $d \leq n$ (Lemma 8.17). We assume that V and G are specified succinctly by the tuple

$$(V, G) := (n, m; (\lambda^1, m(\lambda^1)); \dots; (\lambda^s, m(\lambda^s))) \quad (26)$$

that specifies n and m in unary and the multiplicity $m(\lambda^j)$ (in unary) of each Weyl module $V_{\lambda^j}(G)$ that occurs in the decomposition (25) with nonzero multiplicity. For each copy of $V_{\lambda}(G)$ that occurs in this decomposition, fix the standard monomial basis of $V_{\lambda}(G)$ as defined in [DEP1, LR]. It will be reviewed in Section 8.3 below. This yields a basis $B(V)$ of V , which we call the *standard monomial basis* of V . Let v_1, \dots, v_n be the coordinates of V in this basis.

Let $K[V]^G \subseteq K[V]$ be the ring of invariants, and $V/G := \text{spec}(K[V]^G)$, the categorical quotient [MFK].

Theorem 8.1 (Derksen) (*cf. Theorem 1.1, Proposition 1.2 and Example 2.1 in [D]*) *The ring $K[V]^G$ is generated by homogeneous invariants of degree $\leq l = nm^2d^{2m^2}$.*

This bound is $\text{poly}(n)$ when m is constant, since $d \leq n$ (Lemma 8.17).

Let $K[V]_l^G \subseteq K[V]^G$ be the subspace of invariants of degree l , and $K[V]_{\leq l}^G$ the subspace of non-constant invariants of degree $\leq l$. The spaces $K[V]_l$ and $K[V]_{\leq l}$ are defined similarly. The dimension t of $K[V]_{\leq l}^G$ is bounded by $\dim(K[V]_{\leq l}) = \sum_{c \leq l} \binom{c+n-1}{n-1}$. This is exponential in n , even when m is constant. This worst case upper bound on t is not tight. But we can not expect a significantly better bound since, the singularities of $K[V]^G$ being rational [Bt], the function $h(l) = \dim(K[V]_l^G)$ is, by [F], a quasi-polynomial of degree $\dim(V/G) \geq \dim(V) - \dim(G) = n - m^2$.

Let $F = \{f_1, \dots, f_t\}$ be a set of non-constant homogeneous invariants that form a basis of $K[V]_{\leq l}^G$. By Theorem 8.1, F generates $K[V]^G$. Consider the morphism $\pi_{V/G}$ from V to K^t given by

$$\pi_{V/G} : v \rightarrow (f_1(v), \dots, f_t(v)). \quad (27)$$

By Theorem 2.9, the image of this morphism is closed, and V/G can be identified with this closed image. Let $z = (z_1, \dots, z_t)$ be the coordinates of K^t , I the ideal of V/G under this embedding, $K[V/G]$ its coordinate ring. Then $K[V/G] = K[z]/I$, and we have the comorphism $\pi_{V/G}^* : K[V/G] \rightarrow K[V]$ given by

$$\pi_{V/G}^*(z_i) = f_i. \quad (28)$$

Since f_i 's are homogeneous, $K[V/G]$ is a graded ring, with the grading given by $\deg(z_i) = \deg(f_i)$. Furthermore, $\pi_{V/G}^*$ gives the isomorphism between $K[V/G]$ and $K[V]^G$:

$$\pi_{V/G}^*(K[V/G]) = K[V]^G.$$

8.1 Construction of an h.s.o.p.

Gröbner basis theory implies the following result.

Proposition 8.2 *The problem of constructing an h.s.o.p. for $K[V]^G$ belongs to EXPSPACE.*

The algorithm here needs exponential space and double exponential time even when m is constant because, as already observed, the dimension t of the ambient space K^t containing V/G , cf. eq. (27), is exponential in n even when m is constant. Furthermore, the time requirement of this algorithm remains double exponential even if we only want to construct an S of poly(n) size (not necessarily optimal) such that $K[V]^G$ is integral over the subring generated by S .

If the second fundamental theorem (SFT) akin to that for matrix invariants (Theorem 2.8) holds for $K[V]^G$ (cf. Definition 10.8) then the problem of constructing an h.s.o.p. can be put in $REXP^{NP}$ assuming GRH; cf. Theorem 10.7.

It follows from Sturmfels [Stm2] (cf. Theorem 4.7.1. therein) and Lemma 2.10 that $K[V]^G$ is integral over the subring generated by any homogeneous basis $F' = \{f'_1, \dots, f'_{t'}\}$ of $K[V]_{\leq t'}^G$, $t' = m^2(dm+1)^{m^2}$. We can use F' instead of F in the algorithms of this section. If $V = V_{(d)}(G)$, then $n = \dim(V) = \binom{d+m-1}{m-1}$, and t' is still exponential in n for constant m . So the algorithm in Proposition 8.2 will still need exponential space and double exponential time for constant m .

For the proof of Proposition 8.2 we need the following result.

Lemma 8.3 *Let $F = \{f_1, \dots, f_t\}$ be the set of generators of $K[V]^G$ as in (27). A generating set of syzygies among the elements of F can be constructed in work-space exponential in n and m and time that is double exponential in n and m .*

Proof:

Let $\pi_{V/G}$ be the morphism from V to K^t based on F as in e.q. (27). Using this embedding of V/G and the Gröbner basis algorithm in [MR2], we can compute a generating set of syzygies among the elements of F . This algorithm works in space that is exponential in $\dim(V/G) \leq n$, polynomial in the dimension t of the ambient space, and poly-logarithmic in the maximum degree of the elements in F ; cf. Theorem 1 in [MR2]. This work-space requirement is clearly single exponential in n and m (since $d \leq n$). Q.E.D.

Proof of Proposition 8.2:

This is proved just like the first statement of Proposition 3.1 using Lemma 8.3 instead of Theorem 2.8. Q.E.D.

For constant m the proof of Proposition 2.12 can be constructivized (cf. the proof of Proposition 3.2), using F as in eq. (27), to construct a separating S of cardinality $\leq 2nl = \text{poly}(n)$, where l is as in Theorem 8.1. This algorithm also takes space that is exponential in n and time that is double exponential.

8.2 On derandomization of Noether's normalization lemma

The following result (Theorem 8.5 (2)) says that the double exponential time bound in Proposition 8.2 can be brought down to quasi-polynomial for constant m (even for the strong form of NNL) requiring $|S|$ to be $O(\text{poly}(n))$ (instead of optimal). Before we can state the result, we need a few definitions.

We say that Noether's normalization lemma for $K[V]^G$ is *derandomized* if there exists an *explicit system of parameters* (e.s.o.p) for $K[V]^G$ as defined below. We say that it is derandomized in a strong sense if there exists a separating e.s.o.p. for $K[V]^G$.

Definition 8.4 (a) We say that $S \subseteq K[V]^G$ is an s.s.o.p. (small system of parameters) for $K[V]^G$ if (1) $K[V]^G$ is integral over the subring generated by S , (2) the cardinality of S is $\text{poly}(n, m)$, (3) every invariant in s is homogeneous of $\text{poly}(n, m)$ degree, and (4) every $s \in S$ has a small specification in the form of a weakly skew straight-line program [MP] of $\text{poly}(n, m)$ bit-length over \mathbb{Q} and the coordinates v_1, \dots, v_n of V in the standard monomial basis.

(b) We say that a subset $S \subseteq K[V]^G$ is an e.s.o.p. (explicit system of parameters) for $K[V]^G$ if (1) S is an s.s.o.p. for $K[V]^G$, and (2) the specification of S , consisting of a weakly skew straight-line program as above for each $s \in S$, can be computed in $\text{poly}(n, m)$ time, given n, m , and the nonzero multiplicities $m(\lambda)$'s of $V_\lambda(G)$'s as in eq.(25).

(c) Quasi-s.s.o.p. and quasi-e.s.o.p. are defined similarly by replacing the $\text{poly}(n, m)$ bounds by $O(2^{\text{polylog}(n, m)})$ bounds. For a constant m , subexponential-s.s.o.p. and subexponential-e.s.o.p. with exponent $\delta > 0$ are defined by replacing the $\text{poly}(n, m)$ bounds by $O(2^{O(n^\delta)})$ bounds.

(d) S.s.o.p., e.s.o.p., and the related notions are defined in a relaxed sense by dropping the weakly skew requirement in (a) (4) and the degree requirement in (a) (3).

Here (4) in (a) is equivalent [MP] to (4)': every $s \in S$ can be expressed as the determinant of a matrix of $\text{poly}(n, m)$ size whose entries are (possibly non-homogeneous) linear combinations of v_1, \dots, v_n with rational coefficients of $\text{poly}(n, m)$ bit length. By [Cs], it follows that, given such a weakly-skew straight-line program of an invariant $s \in S$ and the coordinates of any rational point $v \in V$, the value $s(v)$ can be computed in time that is polynomial in n, m , and the total bit-length $\langle v \rangle$ of the specification of v , and even fast in parallel, by a uniform AC^0 circuit of $\text{poly}(n, m, \langle v \rangle)$ bit-size with oracle access to DET .

The following is the full statement of Theorem 1.3.

Theorem 8.5

(1) Suppose the black-box derandomization hypothesis for PIT for diagonal depth three circuits over K holds (cf. Section 2.1). Then $K[V]^G$ has a separating e.s.o.p., if m is constant.

Specifically, there exists a set $S \subseteq K[V]^G$ of $\text{poly}(N)$ invariants, $N = n^{m^2} d^{m^4}$, such that (1) S is separating, and hence (cf. Theorem 2.11) $K[V]^G$ is integral over its subring generated by S , (2) every invariant in S is homogeneous of $\text{poly}(N)$ degree, (3) every $s \in S$ has a weakly skew straight-line program over \mathbb{Q} and v_1, \dots, v_n of $\text{poly}(N)$ bit-length, and (4) the specification of S , consisting of such a weakly-skew straight-line program of every invariant in S , can be computed in $\text{poly}(N)$ time.

Assuming the parallel black-box derandomization hypothesis for PIT for diagonal depth three circuits (cf. Section 2.1), the specification of S can be computed by a uniform AC^0 circuit of $\text{poly}(N)$ bit-size with oracle access to DET .

(2) Suppose m is constant, or more generally, $O(\text{polylog}(n))$ (as is the case if $m = O(\sqrt{d})$; cf. Lemma 8.17). Then $K[V]^G$ has a separating quasi-e.s.o.p (unconditionally). Furthermore, such a separating quasi-e.s.o.p. can be constructed by a uniform AC^0 circuit of quasi- $\text{poly}(n)$ bit-size with oracle access to DET .

Theorem 1.3 follows from (2).

To prove Theorem 8.5, we first recall some results from standard monomial theory [LR, DEP1, DRS] and prove some lemmas. In what follows, we let $N = n^{m^2} d^{m^4}$ as above.

8.3 The standard monomial basis of V

We now define the standard monomial basis of V mentioned in Definition 8.4 following [DEP1, LR] and prove some lemmas concerning its complexity-theoretic properties.

Let $\bar{G} = GL_m(K)$. Let Z be an $m \times m$ variable matrix. Let $K[Z]$ be the ring generated by the variable entries of Z . Let $K[Z]_d$ denote the degree d part of $K[Z]$. It has commuting left and right actions of \bar{G} , where $(\sigma, \sigma') \in \bar{G} \times \bar{G}$ maps $h(Z) \in K[Z]_d$ to $h(\sigma^t Z \sigma')$. For each partition $\lambda : \lambda_1 \geq \dots \geq \lambda_q > 0$, $q \leq m$, the Weyl module $V_\lambda(\bar{G})$ labelled by λ can be embedded in $K[Z]_d$, $d = |\lambda| = \sum_i \lambda_i$, as follows.

Let (A, B) be a bi-tableau of shape λ . This means both A and B are Young tableau [Fu] of shape λ such that (1) each box of A or B contains a number in $[m] = \{1, \dots, m\}$, (2) all columns of A and B are strictly increasing, and (2) all rows are non-decreasing. Let A_i and B_i denote the i -th column of A and B respectively. With any pair (A_i, B_i) of columns, we associate the minor $Z(A_i, B_i)$ of Z indexed by the row numbers occurring in A_i and the column numbers occurring in B_i . With each bi-tableau (A, B) we associate the monomial in the minors of Z defined by $Z(A, B) := Z(A_1, B_1)Z(A_2, B_2)Z(A_3, B_3) \dots$. We call such a monomial *standard* of shape λ and degree $d = |\lambda|$. It is known [DRS] that the standard monomials of degree d form a basis of $K[Z]_d$. We call it the DRS-basis of $K[Z]_d$ and denote it by $B(Z)_d$.

A standard monomial $Z(A, B)$ is called *canonical* if the column B_i , for each i , just consists of the entries $1, 2, 3, \dots$ in the increasing order. It is easy to see that, for each partition λ , the subspace of $K[Z]$ spanned by the canonical monomials of shape λ is a representation of \bar{G} under its left action on $K[Z]$. It is known [DEP1, LR] that this representation is isomorphic to the Weyl module $V_\lambda(\bar{G})$ of \bar{G} , and that the set of canonical monomials of shape λ form its basis. We refer to it as the *standard monomial basis* of $V_\lambda(\bar{G})$, and denote it by $B_\lambda = B_\lambda(\bar{G})$. Each Weyl module $V_\lambda(G)$ of $G = SL_m(K)$ is also a Weyl module of \bar{G} in a natural way. Hence this

also specifies the standard monomial basis B_λ of $V_\lambda(G)$.

Fix the standard monomial basis B_λ in each copy of $V_\lambda(G)$ in the complete decomposition of V as in (25). This yields a basis $B(V)$ of V , which we call its *standard monomial basis*. It depends on the choice of the decomposition of V (if the multiplicities are greater than one). But this choice does not matter in what follows.

Lemma 8.6 (a) *Given any nonstandard monomial μ of degree d in the minors of Z , the representation of μ in the DRS basis $B(Z)_d$ can be computed in $\text{poly}(d^{m^2})$ time. More strongly, it can be computed by a uniform AC^0 circuit of $\text{poly}(d^{m^2})$ bit-size with oracle access to DET .*

(b) *Consider $K[Z]_d$ as a left \bar{G} -module, where $g \in \bar{G}$ maps $h(Z)$ to $(g \cdot h)(Z) = h(g^t Z)$. Then, given the specifying label (a bi-tableau) of any basis element $b \in B(Z)_d$ and $g \in GL_m(\mathbb{Q})$, the representation of $g \cdot b$ in the DRS-basis $B(Z)_d$ can be computed in $\text{poly}(d^{m^2}, \langle g \rangle)$ time, where $\langle g \rangle$ denotes the bit-length of the specification of g . More strongly, it can be computed by a uniform AC^0 circuit of $\text{poly}(d^{m^2}, \langle g \rangle)$ size with oracle access to DET .*

(c) *Let $V_\lambda(\bar{G})$ be a Weyl module of degree d , and B_λ its standard monomial basis as described above. For any basis element $b \in B_\lambda$, and $g \in GL_m(\mathbb{Q})$, the representation of $g \cdot b$ in the basis B_λ can be computed in $\text{poly}(d^{m^2}, \langle g \rangle)$ time. More strongly, it can be computed by a uniform AC^0 circuit of $\text{poly}(d^{m^2}, \langle g \rangle)$ bit-size with oracle access to DET .*

When m is constant, the $\text{poly}(d^{m^2})$ bound becomes $\text{poly}(d) = \text{poly}(n)$.

Proof:

(a) Let $B'(Z)_d$ denote the usual monomial basis of $K[Z]_d$ consisting of the monomials in the entries z_{ij} of Z of total degree d . The cardinality of $B'(Z)_d$ is equal to the number of monomials of degree d in the m^2 variables z_{ij} . This number is $\binom{d+m^2-1}{m^2-1} = \text{poly}(d^{m^2})$. The cardinality of $B(Z)_d$ is the same. Let \mathcal{A}_d be the matrix for the change of basis so that:

$$B(Z)_d = \mathcal{A}_d B'(Z)_d, \quad \text{and} \quad B'(Z)_d = \mathcal{A}_d^{-1} B(Z)_d. \quad (29)$$

The matrix \mathcal{A}_d can be computed in $\text{poly}(d^{m^2})$ time. For this, observe that each row of \mathcal{A}_d corresponds to the expansion of a standard monomial $b \in B(Z)_d$ in the usual monomial basis $B'(Z)_d$. Since the number of monomials of degree $\leq d$ in the m^2 variable entries of Z is $\text{poly}(d^{m^2})$ and the degree of b is d , this expansion can be computed by a uniform weakly skew [MP] straight-line program of $\text{poly}(d^{m^2})$ bit-size (constructed by induction on d). It follows [MP] that it can also be computed fast in parallel by a uniform AC^0 circuit of $\text{poly}(d^{m^2})$ bit-size with oracle access to DET . This yields the representation of b in the basis $B'(Z)_d$. Thus \mathcal{A}_d can be computed by a uniform AC^0 circuit of $\text{poly}(d^{m^2})$ bit-size with oracle access to DET .

Once \mathcal{A}_d has been computed, \mathcal{A}_d^{-1} can also be computed fast in parallel by a uniform AC^0 circuit of $\text{poly}(d^{m^2})$ bit-size with oracle access to DET .

The standard representation in the basis $B(Z)_d$ of any nonstandard monomial $\mu \in K[Z]_d$ in the minors of Z can now be computed fast in parallel as follows. Let $b(\mu)$ and $b'(\mu)$ be the row vectors of the coefficients of the representations of μ in the bases $B(Z)_d$ and $B'(Z)_d$, respectively. Clearly $b(\mu) = b'(\mu) \mathcal{A}_d^{-1}$. Expand μ fast in parallel (as we expanded b above) to get its representation $b'(\mu)$. Multiply this on the right by \mathcal{A}_d^{-1} fast in parallel to get $b(\mu)$.

(b) First we expand $g \cdot b$ fast in parallel (as above) to get its representation in the usual monomial basis $B'(Z)_d$. The representation in $B(Z)_d$ can now be computed fast in parallel by multiplication on the right by \mathcal{A}_d^{-1} .

(c) This follows from (b) using the concrete realization of $V_\lambda(G)$ described before Lemma 8.6 as the G -submodule of $K[Z]_d$, $d = |\lambda|$, spanned by the canonical standard monomials of shape λ . Q.E.D.

We also note the following for future reference.

Lemma 8.7 (a) *Given any standard monomial $b \in B(Z)_d$ and a generic (variable) $u \in \bar{G}$, the representation of $u \cdot b$ in $B(Z)_d$ can be computed in $\text{poly}(d^{m^2})$ time. More strongly, it can be computed by a uniform AC^0 circuit of $\text{poly}(d^{m^2})$ bit-size with oracle access to DET . The coefficients of this representation are polynomials in the entries of u of degree d .*

(b) *Given any basis element $b \in B_\lambda$ and a generic $u \in \bar{G}$, the representation of $u \cdot b$ in the basis B_λ can be computed in $\text{poly}(d^{m^2})$ time. More strongly, it can be computed by a uniform AC^0 circuit of $\text{poly}(d^{m^2})$ bit-size with oracle access to DET . The coefficients of this representation are polynomials in the entries of u of degree $d = |\lambda|$.*

The proof is similar to that of Lemma 8.6.

8.4 Proof of Theorem 8.5 assuming explicitness of V/G

Let $v = (v_1, \dots, v_n)$ be the coordinates of V in the standard monomial basis $B(V)$ of V . Let $x = (x_1, \dots, x_n)$ be new variables. Let

$$X = \sum_i x_i v_i \in K[V; x], \quad (30)$$

be a generic affine combination of v_i 's. Here $K[V; x]$ denotes the ring obtained by adjoining x_1, \dots, x_n to $K[V] = K[v_1, \dots, v_n]$. Then, for any $0 < c \leq l$, $X^c \in K[V; x]$ is a generic linear combination of all monomials in v_i 's of total degree c :

$$X^c = \sum_{a_1, \dots, a_n \geq 0: \sum a_i = c} \binom{c}{a_1, \dots, a_n} \left(\prod_{i \geq 1} x_i^{a_i} \right) \left(\prod_{i \geq 1} v_i^{a_i} \right). \quad (31)$$

Here $\binom{c}{a_1, \dots, a_n}$ denotes the multinomial coefficient, and the monomials $(\prod_{i \geq 1} v_i^{a_i})$ occurring in this expression form a basis of the subspace $K[V]_c \subseteq K[V]$ of polynomials on V of degree c .

Let $R = R_G : K[V] \rightarrow K[V]^G$ denote the Reynolds' operator for G (cf. Section 2.2.1 in [DK]). We denote the induced map from $K[V; x]$ to $K[V]^G[x]$ by R as well. Here $K[V]^G[x]$ denotes the ring obtained by adjoining x_1, \dots, x_n to $K[V]^G$. Now consider a generic invariant

$$R(X^c)(v, x) = \sum_{a_1, \dots, a_n \geq 0: \sum a_i = c} \binom{c}{a_1, \dots, a_n} R\left(\prod_{i \geq 1} v_i^{a_i}\right) \left(\prod_{i \geq 1} x_i^{a_i}\right) \in K[V]^G[x]. \quad (32)$$

Since the monomials $(\prod_{i \geq 1} v_i^{a_i})$ in (31) form a basis of $K[V]_c$, it follows from the properties of the Reynold's operator that the elements $R(\prod_{i \geq 1} v_i^{a_i}) \in K[V]^G$ occurring in (32) span the

subspace $K[V]_c^G \subseteq K[V]^G$ of invariants of degree c . By Theorem 8.1, the invariants of degree $\leq l$ generate $K[V]^G$. Hence the set

$$F = \{R(\prod_{i \geq 1} v_i^{a_i}) \mid \sum_i a_i = c, 0 < c \leq l\}. \quad (33)$$

generates $K[V]^G$.

Let $\Delta_3[n, l, k]$ denote the class of diagonal depth three circuits (cf. Section 2.1) over K and the variables x_1, \dots, x_n with total degree $\leq l$ and top fan-in $\leq k$. The size of any such circuit is $O(knl)$.

The following result is the key to the proof of Theorem 8.5.

Theorem 8.8 *Let $N = n^{m^2} d^{m^4}$, and let $l = nm^2 d^{2m^2}$ as in Theorem 8.1. Given n, m , $0 < c \leq l$, and the specification $\langle V, G \rangle$ of V and G as in eq.(26), one can compute in $\text{poly}(N)$ time the specification of an arithmetic constant depth circuit $C = C[V, m, c]$ over \mathbb{Q} such that (1) C computes the polynomial $R(X^c)(v, x)$ in $x = (x_1, \dots, x_n)$ and $v = (v_1, \dots, v_n)$, and (2) for any fixed $h \in V$, the circuit C_h obtained by specializing the variables v_i 's in C to the coordinates of h in the standard monomial basis $B(V)$ of V is a diagonal depth three circuit in the class $\Delta_3[n, c, k]$, with $k = \text{poly}(N)$.*

More strongly, C can be computed by a uniform AC^0 circuit of $\text{poly}(N)$ bit-size with oracle access to DET .

This result says that V/G for constant m is an explicit variety in the following sense.

Definition 8.9 *We say that $V/G = \text{spec}(K[V]^G)$ is explicit if, given n, m , and the specification $\langle V, G \rangle$ of V and G as in eq.(26), one can compute in $\text{poly}(n, m)$ time a set of arithmetic circuits $C = C[V, m, c]$'s, $1 \leq c \leq q = \text{poly}(n, m)$, over \mathbb{Q} of $\text{poly}(n, m)$ bit size over the variables $x = (x_1, \dots, x_n)$ and $v = (v_1, \dots, v_n)$ such that the polynomials $C[V, m, c](x, v)$'s computed by $C[V, m, c]$'s are of $\text{poly}(n, m)$ degree and can be expressed in the form*

$$C[V, m, c](x, v) = \sum_j f_{j,c}(x) g_{j,c}(v),$$

with homogeneous $f_{j,c}$'s and $g_{j,c}$'s, so that $K[V]^G$ is generated by $g_{j,c}(v)$'s and $f_{j,c}(x)$'s are linearly independent.

We say that V/G is explicit in a relaxed sense if the degree requirement on $C[V, m, c](x, v)$'s is dropped.

We say that V/G is strongly explicit if the circuits $C[V, m, c]$'s are weakly skew of $\text{poly}(n, m)$ degree.

This definition is a special case of a general definition of an explicit variety that we shall formulate later; cf. Definition 10.2.

By Theorem 8.8, V/G is strongly explicit if m is constant, and quasi-strongly-explicit if $m = \text{polylog}(n)$. By Lemma 3.7, V/G in Theorem 3.6 is strongly explicit. More generally:

Conjecture 8.10 *The categorical quotient V/G for V and G as in eq. (25) is explicit in a relaxed sense for any m .*

Before proving Theorem 8.8, let us prove Theorem 8.5 using it.

Proof of Theorem 8.5 (1) assuming Theorem 8.8:

Let N , $k = \text{poly}(N)$, and l be as in Theorem 8.8. Consider the class $\Delta_3[n, l, 2k]$ of diagonal depth three circuits.

By our black-box derandomization assumption for diagonal depth three circuits over K , there exists a hitting set T against $\Delta_3[n, l, 2k]$ that can be computed in $\text{poly}(n, k, l) = \text{poly}(N)$ time. Assuming the parallel black-box derandomization hypothesis, T can be computed by a uniform AC^0 circuit of $\text{poly}(N)$ bit-size with oracle access to DET .

Fix such a T . By the definition of the hitting set, for any circuit $D \in \Delta_3[n, l, 2k]$ such that $D(x)$ is not an identically zero polynomial, there exists $b \in T$ such that $D(b) \neq 0$.

For any $b \in T$ and $0 < c \leq l$, define the invariant

$$r_{b,c} := R(X^c)(v, b) \in K[V]^G,$$

and let

$$S = \{r_{b,c} \mid b \in T, 0 < c \leq l\} \subseteq K[V]^G. \quad (34)$$

The elements of S are homogeneous polynomials in v of degree $\leq l$, which is $\text{poly}(n)$ if m is constant.

Claim 8.11 *The set S is separating.*

Proof: Let w_1, \dots, w_n be auxiliary variables. For every $c \leq l$, define the symbolic difference

$$\tilde{R}^c(x, v, w) = R(X^c)(v, x) - R(X^c)(w, x),$$

where $R(X^c)(w, x)$ is defined just like $R(X^c)(v, x)$ substituting w for v . Suppose $e, f \in V$ are two points such that $r(e) \neq r(f)$ for some $r \in K[V]^G$. It follows that some generator in the set F in eq. (33) assumes different values at e and f . From eq. (32), it follows that, for some $c \leq l$, $\tilde{R}^c(x, e, f)$ is not an identically zero polynomial in x . By Theorem 8.8, $R(X^c)(e, x)$ is computed by a diagonal depth three circuit in the class $\Delta_3[n, l, k]$. Hence $\tilde{R}^c(x, e, f)$ is computed by a diagonal depth three circuit in the class $\Delta_3[n, l, 2k]$. This means, for some $b \in T$, $\tilde{R}^c(b, e, f) \neq 0$. That is,

$$r_{b,c}(e) = R(X^c)(e, b) \neq R(X^c)(f, b) = r_{b,c}(f).$$

This implies that S is separating. This proves the claim.

It follows from the claim and Theorem 2.11 that $K[V]^G$ is integral over the subring generated by S .

For any $b \in T$ and $0 < c \leq l$, let $D_{b,c}$ be the circuit obtained by specializing the circuit $C[V, m, c]$ in Theorem 8.8 at $x = b$. Then $D_{b,c}$ computes $r_{b,c} = R(X^c)(v, b)$ as a polynomial in v . We specify S by giving, for every invariant $r_{b,c} \in S$, the specification of $D_{b,c}$. By Theorem 8.8,

the circuit $D_{b,c}$ has constant depth and $\text{poly}(N)$ bit-size. Hence, it can also be specified by a weakly skew straight-line program of $\text{poly}(N)$ bit size.

By our black-box derandomization hypothesis, the specification of T can be computed in $\text{poly}(N)$ time. Once T is computed, using Theorem 8.8, we can compute in $\text{poly}(N)$ time, for each $b \in T$ and $c \leq l$, the specification of the circuit $D_{b,c}$ computing the invariant $r_{b,c} \in S$. Thus the specification of S in the form of a circuit $D_{b,c}$ for each $r_{b,c}$, or the corresponding weakly skew straight-line program, can be computed in $\text{poly}(N)$ time. Hence, S is a separating e.s.o.p.

Assuming the parallel black-box derandomization hypothesis, T , and hence S , can be computed by a uniform AC^0 circuit of $\text{poly}(N)$ bit-size with oracle access to DET .

This completes the proof of theorem 8.5 (1) assuming Theorem 8.8.

Proof of Theorem 8.5 (2) assuming Theorem 8.8:

By Theorem 2.4, the black-box derandomization hypothesis holds for diagonal depth three circuits allowing a quasi-prefix. Hence, Theorem 8.5 (2) follows from (the proof of) Theorem 8.5 (1) inserting quasi-prefixes in appropriate places.

This completes the proof of Theorem 8.5 assuming Theorem 8.8.

8.5 Explicitness of V/G for constant m

It remains to prove Theorem 8.8. Towards that end, we first prove some auxiliary lemmas.

First we turn to the computation of the Reynolds operator $R = R_G$, $G = SL_m(K)$. Consider the representation morphism $\psi : V \times G \rightarrow V$ given by: $(v, \sigma) \rightarrow \sigma^{-1}v$. Let $\psi^* : K[V] \rightarrow K[V] \otimes K[G]$ denote the corresponding comorphism. Given $f \in K[V]$, let $\psi^*(f) = \sum_i g_i \otimes h_i$, where $g_i \in K[V]$ and $h_i \in K[G]$.

Lemma 8.12 (cf. Proposition 4.5.9 in [DK])

$$R_G(f) = \sum_i g_i R_G(h_i).$$

This reduces the computation of R_G on $K[V]$ to the computation of R_G on $K[G]$. Since G , as an affine variety, has just one G -orbit, R_G maps $K[G]$ to $K[G]^G = K$. Let Z be an $m \times m$ variable matrix. Computation of R_G on $K[G]$ can be reduced to the computation of R_G on $K[Z]$. This is because $K[G] = K[Z]/J$, where J is the principal ideal generated by $\det(Z) - 1$. If $g \in K[G]$ is represented by $f \in K[Z]$ then $R_G(g) = R_G(f) + J$. Furthermore, $K[Z]^G = K[\det(Z), \det(Z)^{-1}]$.

The computation of R_G on $K[Z]$ can be done using Cayley's Ω process as in [H12]. Here Ω is a differential operator on $K[Z]$ defined as follows. For any $h(Z) \in K[Z]$,

$$\Omega(h(Z)) = \sum_{\pi \in S_m} \text{sign}(\pi) \frac{\partial^m h}{\partial z_{1,\pi_1} \partial z_{2,\pi_2} \cdots \partial z_{m,\pi_m}},$$

where S_m is the symmetric group on m letters and π ranges over all permutations of m letters.

Lemma 8.13 (cf. Proposition 4.5.27 in [DK]) Suppose $f \in K[Z]$ is homogeneous. If the degree of f is mr then

$$R_G(f / \det(Z)^p) = \det(Z)^{r-p} \frac{\Omega^r f}{c_{r,m}},$$

where $c_{r,m} = \Omega^r(\det(Z)^r) \in \mathbb{Z}$. If the degree of f is not divisible by m , then $R_G(f) = 0$.

If $g \in K[G]$ is represented by $f \in K[Z]$, then $R_G(g) = \frac{\Omega^r f}{c_{r,m}}$, if the degree of f is divisible by m , and $R_G(g) = 0$ otherwise.

Write $\det(Z)^r = \sum_{\alpha} a_{\alpha} \alpha(z_{1,1}, \dots, z_{m,m})$, where α ranges over the monomials in $z_{i,j}$'s of degree mr , and $a_{\alpha} \in \mathbb{Z}$. Then

$$\Omega^r = \sum_{\alpha} a_{\alpha} \alpha\left(\frac{\partial}{\partial z_{1,1}}, \dots, \frac{\partial}{\partial z_{m,m}}\right).$$

The number of α 's here is $\binom{mr+m^2-1}{m^2-1} = \text{poly}(\deg(f)^{m^2})$, when $\deg(f) = mr$, and the bit-length of each a_{α} is $\text{poly}(m, r) = \text{poly}(\deg(f))$. Hence $\frac{\Omega^r f}{c_{r,m}} \in \mathbb{Q}$, for $f \in \mathbb{Q}[Z]$ of degree mr , can be computed in $\text{poly}(\deg(f)^{m^2}, \langle f \rangle)$ time, where $\langle f \rangle$ denotes the total bit-length of the coefficients of f . This can also be done fast in parallel. Thus:

Corollary 8.14 Given $g \in \mathbb{Q}[G] \subseteq K[G]$ represented as a polynomial $f \in \mathbb{Q}[Z, \det(Z)^{-1}]$, $R_G(g) \in \mathbb{Q}$ can be computed in $\text{poly}(\deg(f)^{m^2}, \langle f \rangle)$ time. More strongly, this can be done by a uniform AC^0 circuit of $\text{poly}(\deg(f)^{m^2}, \langle f \rangle)$ bit-size.

Now let $\bar{G} = GL_m(K)$. Then V as in (25) is also a polynomial \bar{G} -representation in a natural way so that, as a \bar{G} -module:

$$V = \oplus_{\lambda} m(\lambda) V_{\lambda}(\bar{G}). \quad (35)$$

Let $u \in \bar{G}$ be a generic (variable) matrix. Let $0 < c \leq l = \text{poly}(n, d^{m^2})$ and $N = n^{m^2} d^{m^4}$ be as in Theorem 8.8. For X is as in (30), $u \cdot X$ can be expressed as:

$$u \cdot X = \sum_i x_i(u \cdot v_i) = \sum_i e_i(x, u) v_i, \quad (36)$$

where $e_i \in \mathbb{Q}[x, u]$ is a polynomial in x_j 's and the entries of u that is linear in x_j 's and has total degree $\leq d$ in the entries of u , since V is a representation of \bar{G} of degree $\leq d$. Hence,

$$u \cdot X^c = (u \cdot X)^c = \sum_{\mu} \mu \beta_{\mu}(v, x), \quad (37)$$

where μ ranges over the monomials in the entries of u of total degree at most $dc \leq dl = \text{poly}(n, d^{m^2})$ and $\beta_{\mu}(v, x)$ is a polynomial of degree c in $v = (v_1, \dots, v_n)$ as well as $x = (x_1, \dots, x_n)$. The number of μ 's here is $\leq \binom{dc+m^2-1}{m^2-1} = \text{poly}(N)$.

Lemma 8.15 Given n, m, d, c as above, $N = n^{m^2} d^{m^4}$, and the specification $\langle V, G \rangle$ of V and G as in (26), one can compute in $\text{poly}(N)$ time, and more strongly, by a uniform AC^0 circuit of

poly(N) bit-size with oracle access to DET, the specification of a circuit C' over \mathbb{Q} of $\text{poly}(N)$ bit-size on the input variables v_1, \dots, v_n and x_1, \dots, x_n and with multiple outputs that compute the polynomials $\beta_\mu(v, x)$'s in (37). The top (output) gates of C' are all addition gates. Furthermore, for any fixed $h \in V$, the circuit C'_h obtained from C' by specializing the variables v_i 's to the coordinates of h (in the standard monomial basis of V) is a diagonal depth three circuit with multiple outputs in the class $\Delta_3[n, c, e]$, $e = \text{poly}(N)$. By this, we mean that the sub-circuit of C' below each output gate is in $\Delta_3[n, c, e]$.

Proof: We cannot compute $\beta_\mu(v, x)$ in eq.(37) by expanding $(u \cdot X)^c$ as a polynomial in x , u , and v , since the number of terms in this expansion is exponential in n . But we can compute it by a constant depth circuit by evaluating $(u \cdot X)^c$ at several values of u and then performing multivariate Van-der-Monde interpolation in the spirit of [Str1] as follows.

First we show how to construct, for any fixed $g \in GL_m(\mathbb{Q})$, a constant depth circuit A_g that computes the polynomial in v and x given by

$$g \cdot X^c = (g \cdot X)^c = \left(\sum_i e_i(x, g) v_i \right)^c, \quad (38)$$

where $e_i(x, g)$ is a linear form in x that is obtained by evaluating $e_i(x, u)$ in eq.(36) at $u = g$. Towards this end, we first construct a depth two circuit A'_g with addition gate at the top that computes the quadratic polynomial in v and x

$$g \cdot X = \sum_i e_i(x, g) v_i \quad (39)$$

obtained by instantiating (36) at $u = g$. Recall that v_1, \dots, v_n are the coordinates of V corresponding to the standard monomial basis $B(V)$ of V compatible with the decomposition (35). Hence, using Lemma 8.6 (c), the coefficients of the linear form $e_i(x, g)$, for given $g \in GL_m(\mathbb{Q})$, can be computed in $\text{poly}(n, d^{m^2}, \langle g \rangle)$ time, and more strongly, by a uniform AC^0 circuit of $\text{poly}(n, d^{m^2}, \langle g \rangle)$ bit-size with oracle access to *DET*. After this, the specification of A'_g can also be computed in $\text{poly}(n, d^{m^2}, \langle g \rangle)$ time, and more strongly, by a uniform AC^0 circuit of $\text{poly}(n, d^{m^2}, \langle g \rangle)$ bit-size with oracle access to *DET*.

Next we construct A_g with a single multiplication gate of fan-in c at its top that computes the c -th power of $g \cdot X$ computed by the output node of A'_g . The polynomial $A_g(v, x)$ computed by A_g is $(g \cdot X^c)(v, x)$. Furthermore, for any fixed $h \in V$, the circuit obtained by instantiating A_g at $v = h$ is a depth two circuit with multiplication (powering) gate at the top.

Next we show how to efficiently construct a circuit C' for computing the polynomials β_μ 's using A_g 's for several g 's of $\text{poly}(N)$ bit-length.

Let e be the number of monomials μ 's in $u_{i,j}$ with the degree in each $u_{i,j}$ at most $d' = dc$. Then $e = O((dc)^{m^2}) = \text{poly}(N)$, since $c \leq l = \text{poly}(n, d^{m^2})$. Order these monomials lexicographically. For $r \leq e$, let μ_r denote the r -th monomial in this order. Choose $m \times m$ non-negative integer matrices g_1, \dots, g_e such that (1) the $e \times e$ matrix $B = [\mu_r(g_s)]$, whose (s, r) -th entry, for $s, r \leq e$, is $\mu_r(g_s)$, is non-singular, and (2) every entry of each g_s is $\leq d'$. We can choose such g_s 's explicitly so that B is a multivariate Van-der-Monde matrix as described in Section 3.9 in [MP]. Specifically, let $E = [d']^{m^2}$ be the set of e integral points in \mathbb{Z}^{m^2} , where

$[d'] = \{0, \dots, d'\}$. Order E lexicographically. Let g_s be the s -th point in E interpreted as an $m \times m$ matrix. Then B is a non-singular multivariate Van-der-Monde matrix (cf. Sections 3.9 and 3.11 in [MP]). It can be computed in $\text{poly}(N)$ time, and more strongly, by a uniform AC^0 circuit of $\text{poly}(N)$ bit-size. Its inverse B^{-1} can be computed by a uniform AC^0 circuit of $\text{poly}(N)$ bit-size with oracle access to DET .

Let $\bar{\beta}$ denote the column-vector of length e whose r -th entry, for $r \leq e$, is $\beta_{\mu_r}(v, x)$. Let \bar{A} denote the column vector of length e whose s -th entry, for $s \leq e$, is $A_{g_s}(v, x) = (g_s \cdot X^c)(v, x)$. Then, by (37),

$$\bar{A} = B\bar{\beta} \quad \text{and} \quad \bar{\beta} = B^{-1}\bar{A}.$$

Using the second equation, we can construct a constant depth circuit C' (with multiple outputs) for computing the entries of $\bar{\beta}$ using the constant depth circuits A_{g_s} 's constructed above. Each output gate of C' is an addition gate with fan-in $e = \text{poly}(N)$. Each gate at the second level from the top is the c -th powering gate, because the top gate of each A_{g_s} is the c -th powering gate. For a fixed $h \in V$, the circuit C'_h obtained by instantiating C' at $v = h$ is thus a diagonal depth three circuit with multiple outputs in the class $\Delta_3[n, c, e]$.

Since A_{g_s} , for every $g_s \in E$, and B^{-1} can be constructed in $\text{poly}(N)$ time, the construction of C' takes $\text{poly}(N)$ time. More strongly, it can be computed by a uniform AC^0 circuit of $\text{poly}(N)$ bit-size with oracle access to DET . Q.E.D.

Now we turn to the construction of the circuit $C = C[V, m, c]$ for computing $R(X^c)$, as required in Theorem 8.8, given n, d, m, c and the specification $\langle V, G \rangle$ of V and G as in (26).

Let u_{ij} denote the (i, j) -th entry of the generic $u \in \bar{G}$, and $u_{i,j}^{-1}$, the (i, j) -th entry of $u^{-1} = \text{Adj}(u)/\det(u)$. Substituting u^{-1} for u in eq.(37), we get

$$u^{-1} \cdot X^c = (u^{-1} \cdot X)^c = \sum_{\mu} \mu' \beta_{\mu}(v, x), \quad (40)$$

where μ ranges as in eq.(37), and $\mu' \in \mathbb{Q}[u, \det(u)^{-1}]$ is a polynomial in the entries of u and $\det(u)^{-1}$ obtained from μ by substituting $u_{i,j}^{-1}$ for $u_{i,j}$. The degree of the numerator of μ' is again $\text{poly}(n, d^{m^2})$.

By Lemma 8.12 and eq.(40),

$$R(X^c)(v, x) = \sum_{\mu} R_G(\mu') \beta_{\mu}(v, x).$$

Here $R_G(\mu')$ is a rational number that can be computed in $\text{poly}(N)$ time using Corollary 8.14, since the degree of μ' is $\text{poly}(n, d^{m^2})$. Let C' be the circuit for computing β_{μ} 's as in Lemma 8.15. The circuit C is obtained by adding a single addition gate that performs linear combinations of the various output nodes of C' computing β_{μ} 's, the coefficients in the linear combination being the $\text{poly}(N)$ -time-computable rational numbers $R_G(\mu')$. Since the top gates of C' are addition gates with fan-in e , we can ensure, by merging the addition gates in the top two levels, that the depth of C is the same as that of C' . The top gate of C after this merging is an addition gate with fan-in $k = e^2 = \text{poly}(N)$.

Given n, d, m, c , and $\langle V, G \rangle$, the specification of C' can be computed in $\text{poly}(N)$ time by Lemma 8.15. After this, the specification of the circuit C as above can also be computed in

$\text{poly}(N)$ time. More strongly, it can be computed by a uniform AC^0 circuit of $\text{poly}(N)$ bit-size with oracle access to DET .

For any fixed $h \in V$, the circuit C_h obtained by specializing the variables v_i 's in C to the coordinates of h is a diagonal depth three circuit in the class $\Delta_3[n, c, k]$, with $k = e^2 = \text{poly}(N)$. This is because, by Lemma 8.15, C'_h is a diagonal depth three circuit with multiple outputs in the class $\Delta_3[n, c, e]$, $e = \text{poly}(N)$.

This completes the proof of Theorem 8.8. Q.E.D.

With this, we have also completed the proof of Theorem 8.5.

We also note down the following consequence of the proof.

Lemma 8.16 *Let V and G be as in (25). A set of generators of $K[V]^G$ can be constructed by a uniform AC^0 circuit of $\text{poly}(d^{m^{O(1)}n})$ bit-size with oracle access to DET .*

Proof: By Theorem 8.1, $K[V]^G$ is generated by $R_G(\mu)$'s where μ ranges over all monomials of degree $\leq d^{m^{O(1)}}$ in the coordinates of the standard basis of V . The number of such monomials is $O(d^{m^{O(1)}n})$. Using Lemmas 8.7, 8.12, and Corollary 8.14, all these $R_G(\mu)$'s can be computed by a uniform AC^0 circuit of $\text{poly}(d^{m^{O(1)}n})$ bit-size with oracle access to DET . Q.E.D.

The following elementary fact was mentioned in Theorem 8.5, and (a) below has been used implicitly throughout this section.

Lemma 8.17 *Let V be as in (25). Then (a) $\dim(V) = n \geq d$, and (b) $n = \Omega(2^{\Omega(m)})$, if $d = \Omega(m^2)$.*

Hence l and N (as in Theorems 8.1 and 8.5) are $O(\text{poly}(n))$, if m is constant, and $O(2^{\text{polylog}(n)})$, if $m = O(\sqrt{d})$.

Proof: It suffices to prove this when $V = V_\lambda(G)$, $\lambda : \lambda_1 \geq \dots \lambda_r > 0$, $r < m$. Note that $\lambda_m = 0$. If $d = |\lambda| = \Omega(m^2)$, $\lambda_1 \geq m' = m/a$, for some constant a .

The dimension of $V_\lambda(G)$ is equal to the number of semi-standard tableau of shape λ with entries in $[m] = \{1, \dots, m\}$ [Fu]. By semi-standard we mean the columns are increasing and the rows are non-decreasing. Let A denote the Young diagram of shape λ .

(a) It suffices to construct d distinct semi-standard tableau of shape λ with entries in $[m]$.

For each location (i, j) in A , let $T_{i,j}$ be the semi-standard tableau whose k -th column, for $k < j$, consists of the entries $1, 2, 3, 4, \dots$ in the increasing order, and for $k \geq j$, consists of the entries $1, 2, \dots, i-1, i+1, i+2, \dots$ in the increasing order. Clearly $T_{i,j}$'s are distinct and semi-standard, and there are d of them.

(b) It suffices to construct $\Omega(2^{m'/2})$ distinct semi-standard tableau of shape λ with entries in $[m] = \{1, \dots, m\}$.

First construct a basic semi-standard tableau T of shape λ by filling the boxes of A as follows. The first column of T contains the entries $1, 2, 3, \dots$ in the increasing order. Suppose $1 < j \leq m' \leq \lambda_1$ and the $(j-1)$ -st column of T has been constructed. Suppose the length of the j -th column of A is q . We construct the j -th column of T as follows. We copy the first q

entries of the constructed $(j - 1)$ -st column in the j -th column of A . Then we read this j -th column from bottom to top and increase the first encountered entry less than m that can be increased without violating semi-standardness. This is always possible if $j \leq m' \leq \lambda_1$. The entries in the columns of A numbered $m' + 1$ to λ_1 are filled in any way so as to ensure that T is semi-standard.

Now suppose m' is even, the odd case being similar. Given any $\sigma : [1, m'/2] \rightarrow \{0, 1\}$, we construct a semi-standard tableau T_σ using T as follows. The columns of T_σ with numbers $m' + 1$ onwards are the same as those of T . The odd numbered columns of T_σ with numbers less than m' are also the same as those of T . The even numbered columns of T_σ with numbers at most m' are constructed as follows. For any $j \in [1, m'/2]$, if $\sigma(j) = 0$, the $2j$ -th column of T_σ is the same as the $2j$ -th column of T . If $\sigma(j) = 1$, the $2j$ -th column of T_σ is obtained by just copying the first q entries of the $(2j - 1)$ -st column of T , where q is the length of the $2j$ -th column of T . It is easy to see that T_σ 's thus constructed are all semi-standard and distinct. Furthermore, there are $2^{m'/2}$ of them. Q.E.D.

8.6 General m

The following is the full statement of Theorem 1.4.

Theorem 8.18 *Suppose K is an algebraically closed field of characteristic zero. Let V as in (2) be a rational representation of $G = SL_m(K)$ of dimension n . Suppose V/G is explicit in a relaxed sense (cf. Conjecture 8.10).*

(a) *Suppose the black-box derandomization hypothesis for PIT over K holds. Then $K[V]^G$ has a separating e.s.o.p. in a relaxed sense (cf. Definition 8.4).*

(b) *Suppose PIT for circuits over K of size $\leq s$ has $O(2^{s^\epsilon})$ -time-computable hitting set for any small constant $\epsilon > 0$. Then $K[V]^G$ has a separating subexponential-e.s.o.p. in a relaxed sense for any exponent $\delta > 0$.*

(c) *A separating s.s.o.p. in a relaxed sense exists for $K[V]^G$ and can be constructed in workspace that is polynomial in n and m (unconditionally). Assuming GRH, it can be constructed by a Σ_3 -algorithm.*

(d) *Analogues of Theorem 1.2 and Theorem 1.1 (d) (2) and (3) also hold in this setting if V/G is explicit (with degree restriction as in Definition 8.9).*

Proof:

(a): The proof is similar to that of Theorem 8.5 (1), using the assumed explicitness of V/G in place of Theorem 8.8 and the black-box derandomization hypothesis for PIT in place of the black-box derandomization hypothesis for diagonal depth three circuits.

(b): The proof of (a) can be modified in a straightforward manner.

(c): Suppose V/G is explicit in a relaxed sense (cf. Definition 8.9). Given any circuit $C[V, m, c](x, v)$ as in Definition 8.9, let $C[V, m, c](x, v')$ denote the circuit obtained from $C[V, m, c]$ by replacing $v = (v_1, \dots, v_n)$ by another tuple $v' = (v'_1, \dots, v'_n)$ of variables. Let $\tilde{C}[V, m, c](x, v, v') = C[V, m, c](x, v) - C[V, m, c](x, v')$. Let T be the hitting set provided by Theorem 2.1 against the class of circuits obtained by letting the v and v' variables of the circuits $\tilde{C}[V, m, c]$'s range

over the coordinates of the points in $V^2 = V \times V$ (with v being of the coordinates of the first copy of V , and v' of the second). We use this new T in place of the old T in the proof of (a). Let $S \subseteq K[V]^G$ be the set of invariants as in eq.(34) using this new T instead. Since we do not have any efficient algorithm to construct the new T , we do not have any efficient algorithm to construct the new S either. Hence the new S will only be a separating s.s.o.p. (in a relaxed sense) instead of a separating e.s.o.p. (in a relaxed sense). It can be constructed in polynomial work space (or by a Σ_3 -algorithm assuming GRH) by the proof technique of Theorem 3.11 using the assumed explicitness of V/G in place of Lemma 3.7, the key to the proof of Theorem 3.11.

(d) If V/G is explicit (with degree restriction), one only needs the black-box derandomization hypothesis for PIT with small degree in (a). The proof of (a) in this case can be extended just as the proof of Theorem 3.6 was extended to prove Theorem 1.2 and Theorem 1.1 (d) (2) and (3). Degree restriction is needed since Theorem 2.2 (a) only holds for PIT of small degree. Q.E.D.

9 Generalizations

In this section we briefly state generalizations of the preceding results to any classical simple algebraic group instead of the special linear group.

Let $G \subseteq SL_m$ (with the standard embedding) be a classical simple algebraic group over an algebraically closed field K of characteristic zero and V its rational representation of dimension n . Let $K[V]^G$ denote the ring of invariants. Since G is reductive, V decomposes as

$$V = \oplus_{\lambda} m(\lambda) V_{\lambda}(G), \quad (41)$$

where λ ranges over the highest weights of G , $V_{\lambda}(G)$ denotes the irreducible Weyl module [Fu] of G labelled by λ , and $m(\lambda)$ its multiplicity. We assume that V and G are specified succinctly by the tuple

$$\langle V, G \rangle := (n, m; (\lambda^1, m(\lambda^1)); \dots; (\lambda^s, m(\lambda^s))) \quad (42)$$

that specifies n and m (in unary), and the multiplicity $m(\lambda^j)$ (in unary) of each Weyl module $V_{\lambda^j}(G)$ that occurs in the decomposition (41) with nonzero multiplicity. For each copy of $V_{\lambda}(G)$ that occurs in this decomposition, fix the monomial basis B_{λ} of $V_{\lambda}(G)$ as defined in [RS]. We refer to it as the RS-basis of $V_{\lambda}(G)$. (We could also have used the standard monomial basis [LR] here. But this would make the calculations below a bit more involved.) This yields a basis $B(V)$ of V , which we call its RS-basis. The elements of B_{λ} are indexed by LS (Lakshmibai-Seshadri)-paths [LR, RS] instead of tableau now. Let e_i, f_i and k_i 's denote the standard generators of the Lie algebra \mathcal{G} of G . Let v_{λ} denote the highest weight vector of $V_{\lambda}(G)$. With every LS-path η (dominated by the highest weight λ), the article [RS] associates a monomial μ_{η} in the generators f_i 's such that $\mu_{\eta}(v_{\lambda})$ is the element of B_{λ} indexed by η .

We can define an e.s.o.p. and the related notions in this general setting very much as in Definition 8.4 using the basis $B(V)$.

Theorem 9.1 *Analogues of Theorems 8.5, and 8.18 also hold for V and G as above, except for the statements therein concerning parallelization of the construction of a separating e.s.o.p. or quasi-e.s.o.p.*

This result also generalizes to direct products of tori and classical algebraic groups.

Proof: (Sketch): Since the proof is very similar to that for the special linear group, we only sketch how to handle the differences. We only sketch how to extend the proof of Theorem 8.5 in this setting, the other case being similar.

Let $I \subseteq K[SL_m]$ denote the ideal so that $K[G] = K[SL_m]/I$. Let U be a variable $m \times m$ matrix. Identify $K[SL_m] = K[U]/(\det(U) - 1)$. Order the entries of U row-wise. Fix a Gröber basis for I with respect to the reverse lexicographic degree ordering on the monomials in the entries of U . Since m is constant such a Gröber basis can be computed in constant time. Let B' be the resulting basis of $K[G]$ consisting of the standard monomials in the variable entries of U . Let $K[G]_{\leq d} \subseteq K[G]$ be the subspace spanned by standard monomials of total degree $\leq d$ in the entries of U . Both $K[G]$ and $K[G]_{\leq d}$ are $G \times G$ -modules, with the first copy of G acting on the left and the second copy on the right. Let $B'_{\leq d} = B' \cap K[G]_{\leq d}$ be the restricted basis of $K[G]_{\leq d}$. We have the Peter-Weyl decomposition [Fu]

$$K[G] = \oplus_{\lambda} V_{\lambda}(G)^* \otimes V_{\lambda}(G), \quad (43)$$

where λ ranges over the highest weights of G . Let

$$B = \oplus_{\lambda} B_{\lambda}^* \otimes B_{\lambda}$$

be the RS-basis of $K[G]$, and let $B_{\leq d} = B \cap K[G]_{\leq d}$ be the restricted basis. The elements of B are indexed by the pairs (ζ^*, η) of LS-paths dominated by (λ^*, λ) for some λ . Let $v_{\lambda^*, \lambda} = v_{\lambda^*}^* \otimes v_{\lambda}$ denote the highest weight vector of the $G \times G$ irreducible module $V_{\lambda}(G)^* \otimes V_{\lambda}(G) \subseteq K[G]$. With any pair (ζ^*, η) of LS-paths dominated by (λ^*, λ) , [RS] associates a unique pair $(\mu_{\zeta^*}, \mu_{\eta})$ of monomials in f_i 's such that the element $b_{\zeta^*, \eta}^{\lambda^*, \lambda}$ of $B_{\lambda^*}^* \otimes B_{\lambda}$ indexed by (ζ^*, η) is $(\mu_{\zeta^*}, \mu_{\eta})v_{\lambda^*, \lambda}$, with μ_{ζ^*} acting on the left and μ_{η} on the right.

The highest weight vectors $v_{\lambda^*, \lambda} \in B_{\leq d}$ satisfy the equations

$$(e_i, e_j)v_{\lambda^*, \lambda} = 0, \quad \text{for all } i, j,$$

where e_i 's acts on the left and e_j 's on the right. Solving this system of equations, and classifying the solutions by weights, we get the specifications of all highest weight vectors $v_{\lambda^*, \lambda} \in B_{\leq d}$ represented in the basis $B'_{\leq d}$. (Here we are using the fact that, for each (λ^*, λ) , the highest weight vector $v_{\lambda^*, \lambda} \in B$ is unique, since $K[G]$ contains only one copy of $V_{\lambda}(G)^* \otimes V_{\lambda}(G)$.) Since the action of (f_i, f_j) 's on $B'_{\leq d}$ is easy to compute, the action of the monomial operators $(\mu_{\zeta^*}, \mu_{\eta})$'s on the basis $B'_{\leq d}$ is also easy to compute by induction on length. Thus we can compute the specifications of $b_{\zeta^*, \eta}^{\lambda^*, \lambda} = (\mu_{\zeta^*}, \mu_{\eta})v_{\lambda^*, \lambda}$'s in $B'_{\leq d}$ efficiently. It can be shown that this can be done in $\text{poly}(d^{m^2})$ time. This yields the transition matrix $T_{\leq d}$ from $B_{\leq d}$ to $B'_{\leq d}$. In additional $\text{poly}(d^{m^2})$ time we can also compute its inverse $T_{\leq d}^{-1}$.

Since $V_{\lambda}(G)$ can be embedded in an appropriate $K[G]_{\leq d}$ via the Peter-Weyl decomposition (43), we can carry out all calculations on $V_{\lambda}(G)$ within $K[G]_{\leq d}$ using the bases $B'_{\leq d}$, $B_{\leq d}$, and the transition matrices $T_{\leq d}$ and $T_{\leq d}^{-1}$. After this the analogues of Lemmas 8.6, 8.7 in this general setting are easy to prove (except for the statements concerning parallelization). The analogue of Corollary 8.14 is also easy to prove. We can not use Cayley's Ω process now as it is specific to the special linear group. But the Reynold's operator on $K[G]$ corresponds to the projection

onto the trivial $G \times G$ -module in the Peter-Weyl decomposition (43) of $K[G]$. So $R_G(w)$, for any $w \in B'_{\leq d}$, can be calculated by first expressing w in the basis $B_{\leq d}$ using the matrix $T_{\leq d}^{-1}$, and then projecting it onto the trivial $G \times G$ -module in the Peter-Weyl decomposition (43) of $K[G]_{\leq d}$. The rest of the proof is similar to the proof in the case of the special linear group. We omit the details. Q.E.D.

The following is the conjectural generalization of Theorem 3.13 (a) in this setting.

Conjecture 9.2 *Suppose V and G are as in Theorem 9.1. Then, given any rational $v \in V$, the instability flag of a one-parameter subgroup driving v to the unique closed G -orbit in the G -orbit closure of v can be computed in time that is polynomial in n , m , and the bit-length of the specification of v .*

10 Equivalence

In this section we describe the relation (Theorems 1.5 and 1.6) between a stronger form of black-box derandomization and derandomization of Noether's normalization lemma for explicit varieties.

10.1 Stronger forms of black box derandomization of PIT

First, we define this stronger form of black-box derandomization of PIT for small degree circuits.

Let K be an algebraically closed field of characteristic zero and $x = (x_1, \dots, x_r)$ a set of r variables. The stronger black-box derandomization problem in this context is to construct in $\text{poly}(s)$ time a *hitting set* against all nonzero polynomials $f(x) \in K[x]$ of degree $\leq d = O(s^a)$, $a > 0$ a constant, that can be approximated infinitesimally closely by arithmetic circuits over K and x of size $\leq s$.

By infinitesimally close approximation, we mean that, given any $\epsilon > 0$, there exists such a circuit $C = C_\epsilon$ of size $\leq s$ such that the distance $\|C(x) - f(x)\|_2$ between the coefficient vectors of $C(x)$ and $f(x)$ in the L_2 -norm is less than ϵ . The circuit C can depend on ϵ . By a hitting set, we mean a set $S_{r,s} \subseteq \mathbb{N}^r$ of test inputs such that (1) the total bit-length of the specification of each test input is $\text{poly}(s)$, and (2) for every nonzero $f(x)$ of degree $\leq d$ that can be approximated infinitesimally closely by circuits over K of size $\leq s$, $S_{r,s}$ contains a test input b such that $f(b) \neq 0$.

The *strong black-box-derandomization hypothesis* for PIT for small degree circuits is that there exists a $\text{poly}(s)$ -time-computable hitting set $S_{r,s}$. The strong black-box derandomization hypothesis for general PIT without any degree restriction is defined similarly. Parallel versions of these hypotheses are defined as in Section 2.1. A similar strong black-box derandomization hypothesis for SDIT (symbolic determinant identity testing), cf. Section 2.1, is that, given m , one can construct in $\text{poly}(m)$ time a hitting set against all nonzero homogeneous polynomials over K of degree m that can be approximated infinitesimally closely by symbolic determinants of size m . The strong black-box derandomization hypothesis for the symbolic permanent identity testing (SPIT) is similar, using the permanent in place of the determinant.

The following result is the analogue of Theorem 2.2 for strong black-box derandomization. This is why PIT is expected to have efficient strong black-box derandomization.

Theorem 10.1 *Suppose K is a field of characteristic zero.*

(a) *Suppose there exists an exponential-time-computable multilinear polynomial p in m variables with integral coefficients of $\text{poly}(m)$ bit-length such that p can not be approximated infinitesimally closely by arithmetic circuits over K of $O(2^{m^a})$ size for some constant $a > 0$. Then PIT for small degree circuits has $O(2^{\text{polylog}(s)})$ -time-computable strong black-box derandomization.*

(b) *If p cannot be approximated infinitesimally closely by arithmetic circuits over K of $O(m^a)$ size for any constant $a > 0$, then PIT for small degree circuits has $O(2^{s^\epsilon})$ -time computable strong black-box derandomization, for any $\epsilon > 0$.*

(c) *Suppose there exists an exponential-time-computable multilinear polynomial p in m variables with integral coefficients of $\text{poly}(m)$ bit-length such that p can not be approximated infinitesimally closely by arithmetic circuits over K , with oracle gates for the permanent, of $O(2^{m^a})$ size for some constant $a > 0$. Then SPIT has $O(2^{\text{polylog}(s)})$ -time-computable strong black-box derandomization.*

(d) *Suppose there is an exponential-time computable multilinear polynomial p in m variables with integral coefficients of $\text{poly}(m)$ bit-length such that p can not be approximated infinitesimally closely by arithmetic circuits over K , with oracle gates for the permanent, of $O(m^a)$ size for any constant $a > 0$. Then SPIT has $O(2^{s^\epsilon})$ -time computable strong black-box derandomization, for any $\epsilon > 0$.*

Proof: (Sketch) The proof is similar to that of Theorem 7.7 in [KI]. Hence we only sketch how to modify the proof in [KI]. We only consider (a), the cases (b), (c), and (d) being similar.

(a) We want to construct a hitting set against every nonzero polynomial $f(x)$ of degree $\leq d = O(s^a)$, $a > 0$ a constant, that can be approximated infinitesimally closely by arithmetic circuits over K of size $\leq s$. Choose $m = (\log s)^e$, for a large enough constant e to be fixed later. With this choice of m and using the hard function $p(y_1, \dots, y_m)$, we get an arithmetic NW-generator NW^p as in the proof of Theorem 7.7 in [KI]. Let H denote the set of test inputs constructed using this NW-generator as in [KI]. We claim that H is a hitting set as desired. Suppose to the contrary that $f(b) = 0$, for every $b \in H$, for some nonzero $f(x)$ of degree $\leq d$ that can be approximated infinitesimally closely by arithmetic circuits over K of size $\leq s$. It then follows as in the proof of Lemma 7.6 in [KI], but using Theorem 2.3 instead of Lemma 7.2 in [KI], that $p(y_1, \dots, y_m)$ can be computed by an arithmetic circuit C over K of size $O(s^c)$ using oracle gates for f for some constant $c > 0$ independent of e . Given any circuit D_δ of size $\leq s$ for approximating f within precision $\delta > 0$, let C_δ denote the circuit obtained from C by substituting D_δ for f . Since f can be approximated infinitesimally closely by circuits of size $\leq s$, by choosing δ small enough, $C_\delta(y)$ can approximate $p(y_1, \dots, y_m)$ to any precision. The size of C_δ is $O(s^{c+1})$. For any δ , choosing e large enough, the size of C_δ can be made $\leq 2^{m^\epsilon}$ for any $\epsilon > 0$. This contradicts hardness of p . Q.E.D.

10.2 Explicit algebraic varieties

Next we define an explicit algebraic variety.

Definition 10.2 Let K be an algebraically closed field of characteristic zero.

(a) A family $\{W_n\}$, $n \rightarrow \infty$, of affine varieties is called *explicit* if there exists a map $\psi_n : K^r \rightarrow K^m$:

$$v = (v_1, \dots, v_r) \rightarrow (f_1(v), \dots, f_m(v)), \quad (44)$$

$r = \text{poly}(n)$, $m = n^{\omega(1)}$, $\log m = O(\text{poly}(n))$, each f_j a homogeneous polynomial of $\text{poly}(n)$ degree, and homogeneous polynomials $g_j(x)$, $x = (x_1, \dots, x_n)$, $1 \leq j \leq m$, of $\text{poly}(n)$ degree such that:

1. W_n is the closure of the image $\text{Im}(\psi_n)$ of ψ_n .
2. The polynomial $F_n(v, x) = \sum_j f_j(v)g_j(x)$ is uniformly p -computable [V1]. By uniform, we mean that one can compute in $\text{poly}(n)$ time a $\text{poly}(n)$ -size circuit C_n over \mathbb{Q} that computes $F_n(v, x)$. Its degree $\deg(F_n)$ is $\text{poly}(n)$.
3. The polynomials $g_j(x)$'s are linearly independent.

We call ψ_n the map defining W_n , and F_n the polynomial defining W_n . We specify W_n succinctly by the circuit C_n .

We say that $\{W_n\}$ is *strongly explicit* if the circuit C_n is weakly skew.

(b) We say that $\{W_n\}$ is *weakly explicit*, if $F(v, x)$ above is equal to the permanent of a matrix D_n of $\text{poly}(n)$ dimension such that (1) each entry of D_n is a homogeneous bilinear form in v and x , and (2) the description of D_n is $\text{poly}(n)$ -time computable.

We say that $\{W_n\}$ is *positive* if the coefficients of all bilinear functions that occur as the entries of D_n are all non-negative rational numbers.

(c) A family of projective varieties is called *explicit* (strongly explicit, weakly explicit, or positive) if the family of the affine cones of these varieties is explicit (respectively, strongly explicit, weakly explicit, or positive).

(d) A family explicit affine or projective varieties in a relaxed sense (without any degree restriction) is defined just as in (a) and (c) but without putting any restriction on the degrees of f_j, g_j and F_n .

The definition of an explicit or weakly explicit variety over an algebraically closed field K of arbitrary characteristic is similar.

10.2.1 Examples of explicit varieties

(1) Explicit categorical quotients and related varieties: The variety V/G , with V and G as in Theorem 3.6, is a strongly explicit variety with the defining map $\psi = \pi_{V/G}$ as in (12); cf. Lemma 3.7. Explicitness of V/G is a key ingredient in the proof of Theorem 3.6. The variety V/G for V and G as in Theorem 4.1 is also strongly explicit.

By Theorem 8.8, the variety V/G , with V and G as in Theorem 8.5, is strongly explicit for constant m , and quasi-strongly-explicit for $m = \text{polylog}(n)$. Explicitness of V/G is a key ingredient in the proof of Theorem 8.5. The variety V/G is explicit in a relaxed sense without any degree requirement for any m as per Conjecture 8.10.

Let $U = (U_1, \dots, U_r)$ and $U' = (U'_1, \dots, U'_r)$ be two tuples of $m \times m$ variable matrices. Think of them as the coordinates of $V = M_m(K)^r$. Expand $\tilde{T}_l(X, U, U')$ in eq. (16) as

$$\tilde{T}_l(X, U, U') = \sum_{\alpha} T_{\alpha}(X)(T_{\alpha}(U) - T_{\alpha}(U')) = \sum_{[\alpha]} |[\alpha]| T_{[\alpha]}(X)(T_{[\alpha]}(U) - T_{[\alpha]}(U')), \quad (45)$$

where $[\alpha] = [\alpha_1 \dots \alpha_l]$ ranges over the equivalence classes of all words of length l with each $\alpha_j \in [r]$, and $|[\alpha]|$ denotes the cardinality of $[\alpha]$. Let $Y[V, G, l]$, $l \leq m^2$, be the strongly explicit variety defined by $\tilde{T}_l(X, U, U')$, with $T_{[\alpha]}(U) - T_{[\alpha]}(U')$'s playing the role of f_i 's in Definition 10.2 and $|[\alpha]| T_{[\alpha]}(X)$'s playing the role of g_j 's. The construction of a separating e.s.o.p. for $K[V]^G$ in the proof of Theorem 3.6 is based on explicitness of the varieties $Y[V, G, l]$'s. The variety V/G in Theorem 3.6 has explicit defining equations given by the second fundamental theorem for matrix invariants [Pr, Rz]. Finding similar explicit defining (or close to defining) equations for $Y[V, G, l]$'s seems extremely hard. This is why the problem of constructing a separating e.s.o.p. (unconditionally) turns out to be so wild.

(2) The Grassmanian (and G/P in general) is an explicit variety with the defining map ψ being the well-known Plücker map (cf. (47) below). Explicitness of the Grassmanian is a key ingredient in the black-box derandomization of the Grassmanian SDIT below (Proposition 11.5).

(3) The variety $\Delta[\det, m]$ associated with the determinant in [MS1] (in the context of the permanent vs. determinant problem) is explicit. The variety $\Delta[\text{perm}, n, m]$ associated with the permanent in [MS1] (in the same context) is weakly explicit and positive. These varieties are defined as follows.

Let X be an $m \times m$ variable matrix. Let Y be an $n \times n$ submatrix of X , say its lower-right $n \times n$ subminor. Let z be any entry of X outside Y . Let \mathcal{X} be the vector space over \mathbb{C} of homogeneous polynomials of degree m in the variable entries of X . Thus $g = \det(X)$ is an element of \mathcal{X} . Then \mathcal{X} is a representation of $GL_{m^2}(\mathbb{C})$, where $\sigma \in GL_{m^2}(\mathbb{C})$ maps $h(X) \in \mathcal{X}$ to $h(\sigma^{-1}X)$, thinking of X as an m^2 -vector. Let $P(\mathcal{X})$ be the projective space associated with \mathcal{X} . Then $\Delta[\det, m] \subseteq P(\mathcal{X})$ is the closure of the orbit $Gg \subseteq P(\mathcal{X})$ in the usual complex topology of $P(\mathcal{X})$. The variety $\Delta[\text{perm}, n, m] \subseteq P(\mathcal{X})$ is constructed similarly using the homogeneous polynomial $z^{m-n} \text{perm}(Y) \in \mathcal{X}$ in place of the determinant.

The affine cone of $\Delta[\det, m]$ is explicit with the defining map $\psi : M_{m^2}(K) \rightarrow \mathcal{X}$ that maps $v \in M_{m^2}(K)$ to $\det(vX)$. The polynomial defining $\Delta[\det, m]$ is $\det(vX)$.

The affine cone of $\Delta[\text{perm}, n, m]$ is weakly explicit and positive.

(4) Explicit variety associated with a p -computable polynomial:

Let $\{p_n(v, x)\}$, $v = (v_1, \dots, v_r)$, $x = (x_1, \dots, x_n)$, be a uniform p -computable family of polynomials over K homogeneous in v . Let $p_n(v, x) = \sum_{\mu} f_{\mu}(v) \mu(x)$, where μ ranges over all monomials in x of total degree $\leq \deg(p_n) = \text{poly}(n)$. Let m be the number of such monomials. Let $\psi = \psi_n$ be the map

$$\psi : v \in K^r \rightarrow (\dots, f_{\mu}(v), \dots) \in K^m.$$

Let $g_{\mu}(x) = \mu(x)$. Then $W_n = \overline{\text{Im}(\psi_n)}$ is an explicit variety with the defining map ψ_n and the defining polynomial p_n .

(5) Explicit toric variety associated with a p -computable polynomial:

Let $\{p_n(x)\}$, $x = (x_1, \dots, x_n)$, be a uniform p -computable homogeneous polynomial over x and K . Let $p_n(x) = \sum_{\mu} a_{\mu} \mu(x)$, where $a_{\mu} \in K$ and μ ranges over all monomials in x of total degree $= \deg(p_n) = \text{poly}(n)$. Let m be the number of such monomials. Consider the monomial map ψ_n :

$$\psi_n : v = (v_1, \dots, v_n) \in K^n \rightarrow (\dots, a_{\mu} \mu(v), \dots) \in K^m.$$

Let $W_n = \overline{\text{Im}(\psi_n)}$, and $P(W_n)$ its projectivization. Then $P(W_n)$ is an explicit toric variety, with the defining polynomial

$$F_n(v, x) = \sum_{\mu} a_{\mu} \mu(v) \mu(x),$$

which is p -computable and uniform.

(6) The toric variety in characteristic zero associated with the Birkoff polytope (cf. Section 6.2 in [DS]) is weakly explicit and positive.

(7) We call an explicit W with $\dim(W) = 1$ an explicit curve. We define explicit surfaces, explicit three-folds, and so on, similarly.

10.3 Implication of strong black-box derandomization for explicit varieties

We now describe an implication of strong black-box derandomization for explicit varieties.

Definition 10.3 Let $W = W_n$ be an explicit variety as in Definition 10.2, z_1, \dots, z_m the coordinates of K^m , and ψ^* the comorphism of ψ in (44). Note that $K[W]$ is graded, with $\deg(z_j) = \deg(f_j)$.

(a) We say that $s \in K[W]$ has a short specification if $\psi^*(s)$ has a straight-line program over \mathbb{Q} and v_1, \dots, v_r of $O(\text{poly}(n))$ bit-length that computes the polynomial function on K^r corresponding to $\psi^*(s)$.

(b) We say that a set $S \subseteq K[W]$ is an explicit system of parameters (e.s.o.p.) for $K[W]$ if (1) each element $s \in S$ has a short specification as in (a) and is homogeneous of $\text{poly}(n)$ degree, (2) $K[W]$ is integral over its subring generated by S , (3) the size of S is $\text{poly}(n)$, and (4) the specification of S , consisting of a straight-line program for $\psi^*(s)$ for each $s \in S$ as in (a), can be computed in $\text{poly}(n)$ time.

An s.s.o.p. is defined by dropping the condition (4).

(c) We say that Noether's normalization lemma for the coordinate ring $K[W]$ of W is derandomized if $K[W]$ has an e.s.o.p.

(d) We say that $s \in K[W]$ is strict if, for some $b \in \mathbb{N}^n$ of $\text{poly}(n)$ bit-length and $0 < c \leq \deg(F_n)$, $s = \sum_j z_j g_j(b)$, where j ranges over all indices such that $\deg(f_j) = c$. Such a strict s can be specified by the pair (b, c) . We call an e.s.o.p. strict, if its each element is strict. A strict e.s.o.p. is specified by the pairs for all its elements. We say that Noether's normalization lemma for $K[W]$ has strict derandomization if $K[W]$ has a strict e.s.o.p. A strict s.s.o.p. is defined similarly.

(e) Strict derandomization of a weakly explicit but positive variety (Definition 10.2) is defined similarly.

(f) An s.s.o.p. or e.s.o.p. (strict or otherwise) and derandomization in a relaxed sense (without degree restriction) are defined similarly by dropping the degree requirement in (b) (1).

For each s in a strict, weakly explicit, positive system S of parameters for $K[W_n]$ as in (e), $\psi^{-1}(s)$ is the permanent of a matrix of polynomial size each whose entries is a linear combination of v_i 's with non-negative rational coefficients. Given any rational non-negative v , whether $\psi^{-1}(s)(v)$ is nonzero can thus be decided in polynomial time and approximate value of $\psi^{-1}(s)(v)$ can also be computed efficiently by an FPRAS [JSV].

The following is a full statement of Theorem 1.5 (3).

Theorem 10.4 *Let K be an algebraically closed field of characteristic zero.*

(a) *Noether's normalization lemma for the coordinate ring $K[W]$ of an explicit variety W has strict derandomization if PIT for small degree circuits over K has strong black box derandomization. (Analogous statement also holds in a relaxed sense without any degree restriction on the explicit variety and PIT.)*

(b) *Noether's normalization lemma for the coordinate ring of the explicit variety $\Delta[\det, m]$ (cf. Example 3 in Section 10.2.1) has strict derandomization if SDIT has strong black box derandomization. The straight-line programs of an e.s.o.p. here can also be assumed to be weakly skew.*

(c) *Noether's normalization lemma for the coordinate ring $K[W]$ of a weakly explicit but positive variety W has strict derandomization if SPIT has strong black box derandomization.*

(d) *The e.s.o.p. S constructed in (a), (b), or (c) can also be assumed to be separating.*

Here we say that S is separating if for any two distinct points $u, v \in W$ there exists an $s \in S$ such that $s(u) \neq s(v)$. We say that Noether's Normalization Lemma for $K[W]$ can be derandomized in a strong form if $K[W]$ has a separating e.s.o.p.

Analogous result holds over an algebraically closed field of any characteristic.

Proof: We will only prove (a) and (b), (c) being similar, and (d) being a simple extension of (a), (b), and (c).

(a) Let $W = W_n$ be an explicit variety as in Definition 10.2, and C_n the circuit computing F_n as there. Let $s = \text{poly}(n)$ be its size, and $d = \text{poly}(n)$ its degree.

By the strong black-box derandomization hypothesis, there exists a $\text{poly}(s)$ -time computable hitting set T against all nonzero polynomials $h(x) = h(x_1, \dots, x_n)$ of degree $\leq d$ that can be approximated infinitesimally closely by arithmetic circuits over K of size $\leq s$.

For each $b \in T$ and $0 < c \leq \deg(F_n)$, define $h_{b,c}(z) := \sum_j z_j g_j(b) \in K[W]$, where j ranges over all indices such that $\deg(f_j) = c$, and $z = (z_1, \dots, z_m)$ denote the coordinates of K^m . Then $\deg(h_{b,c}) = c$. Let

$$S = \{h_{b,c}(z) \mid b \in T, 0 < c \leq \deg(F_n)\} \subseteq K[W]. \quad (46)$$

Since each element in S is clearly strict, it suffices to show that S is an e.s.o.p.; cf. Definition 10.3. This follows from Lemma 10.5 below.

(b) Use Lemma 10.6 instead of Lemma 10.5. Q.E.D.

Lemma 10.5 *Suppose W is an explicit variety and PIT for small degree circuits has strong black box derandomization. Let S and T be as in (46). Then:*

- (a) $W \cap Z(S) = \{0\}$, where $Z(S) \subseteq K^m$ is the zero set of S , and 0 denotes the origin in K^m .
- (b) The coordinate ring $K[W]$ is integral over the subring generated by S .
- (c) The set S is an e.s.o.p.

Proof: Let $\psi = \psi_n$, f_j , g_j , and $F = F_n(v, x)$ be as in Definition 10.2.

(a) Consider any nonzero point $w = (w_1, \dots, w_m) \in W \subseteq K^m$. We have to show that $h_{b,c}(w) \neq 0$ for some $b \in T$ and $0 < c \leq \deg(F_n)$. Let $F_w(x) = \sum_j w_j g_j(x)$. Recall that $K[W]$ is graded, with $\deg(z_j) = \deg(f_j)$. Let $F_w(x)_c$ denote the degree c component of $F_w(x)$. Then $F_w(b)_c = h_{b,c}(w)$. So we have to show that $F_w(b)_c \neq 0$ for some $b \in T$ and $0 < c \leq \deg(F_n)$.

Since $W = \overline{\text{Im}(\psi)}$, there exists, for any $\delta > 0$, an $h_\delta \in K^r$ such that $\|\psi(h_\delta) - w\|_2 \leq \delta/2^{n^c}$, for some large enough positive constant c to be chosen later. Taking δ to be small enough, we can assume that $\psi(h_\delta) \neq 0$. Since $\psi(h_\delta) = (f_1(h_\delta), \dots, f_m(h_\delta))$ and W is explicit, it follows from Definition 10.2 (a) that $F_n(h_\delta, x) = \sum_j f_j(h_\delta) g_j(x)$ is not an identically zero polynomial in x . Let C_n be the circuit computing $F_n(v, x)$ as in Definition 10.2. Let $C_{n,\delta}$ be the circuit obtained from C_n by specializing v to h_δ . Then the size of $C_{n,\delta}$ is $s = \text{size}(C_n) = \text{poly}(n)$ and the degree is $d = \deg(C_n) = \text{poly}(n)$. Furthermore,

$$\|C_{n,\delta}(x) - F_w(x)\|_2 = \left\| \sum_j (f_j(h_\delta) - w_j) g_j(x) \right\|_2 \leq 2^{n^c} \|\psi(h_\delta) - w\|_2 \leq \delta,$$

for some large enough positive constant c . Since, δ can be made arbitrarily small, it follows that $F_w(x)$ can be approximated by circuits of degree $\leq d$ and size $\leq s$. Since T is a hitting set, there exists $b \in T$ such that $F_w(b) \neq 0$. Hence $F_w(b)_c \neq 0$ for some $c \leq \deg(F_n)$. This proves (a).

(b) By (a) and Hilbert's Nullstellensatz [E], it follows that, given any $t \in K[W]$, t^l belongs to the ideal (S) in $K[W]$ generated by S for some large enough positive integer l . Since $K[W]$ is graded, it now follows from the graded Noether's normalization lemma (Lemma 2.10 (a)) that $K[W]$ is integral over its subring generated by S . This proves (b).

(c) We have to verify the properties (1)-(4) in Definition 10.3 (b).

(1) We have to show that each $h_{b,c}(z) \in S$ has a short specification. We have $\psi^*(h_{b,c})(v) = F_n(v, b)_c$. Since W is explicit, cf. Definition 10.2, we can compute the description of the circuit C_n over \mathbb{Q} computing F_n in $\text{poly}(n)$ time. Hence the total size of C_n , including the bit-lengths of the constants in it, is $\text{poly}(n)$. The circuit $C_{n,b}$ for computing $F_n(v, b)$ is obtained by instantiating the circuit C_n at $x = b$. Hence its total size (including the bit-lengths of the constants) is $\text{poly}(n)$. Using Van-der-Monde interpolation as in [Str1] (cf. also the proof of Lemma 8.15 where this technique was used), we can construct using $C_{n,b}$ a circuit $C_{n,b;c}$ of $\text{poly}(n)$ size for computing $\psi^*(h_{b,c})(v) = F_n(v, b)_c$. Thus $C_{n,b;c}$ can be constructed in $\text{poly}(n)$ time. It has $\text{poly}(n)$ bit-length and $\text{poly}(n)$ degree. This specification of $C_{n,b;c}$ can be converted into a straight-line program of $\text{poly}(n)$ bit-length. This shows that each $h_{b,c}(z)$ has a short specification.

(2) It follows from (b) that $K[W]$ is integral over the subring generated by S .

(3) Since the size of T is $\text{poly}(s) = \text{poly}(n)$, and $\deg(F_n)$ is $\text{poly}(n)$, the size of S is clearly $\text{poly}(n)$.

(4) We saw above that the specification of each circuit $C_{n;b;c}$ computing $\psi^*(h_{b,c})$ can be computed in $\text{poly}(n)$ time. Hence it follows that the specification of S , consisting of a circuit $C_{n;b;c}$ computing $\psi^*(h_{b,c})$ for each $h_{b,c} \in S$, can be computed in $\text{poly}(n)$ time.

This shows that S is an e.s.o.p. Q.E.D.

Lemma 10.6 *Suppose $\Delta[\det, m]$ is the explicit variety as in Example 3 in Section 10.2.1. Assume that SDIT has strong black box derandomization.*

Then the set S in (46) is an e.s.o.p. for the coordinate ring of $\Delta[\det, m]$. The straight-line programs of the elements in S can also be assumed to be weakly skew.

Proof: The proof is just like that of Lemma 10.5. We only observe that in the case of $\Delta[\det, m]$ the defining polynomial $F_n(v, x)$ in Definition 10.2 is the determinant of a matrix M_n of $\text{poly}(n)$ size whose each entry is a bilinear function in v and x ; cf. Example 3 in Section 10.2. Furthermore, the specification of M_n can be computed in $\text{poly}(n)$ time. Hence we can use $\det(M_n)$ in place of C_n in the proof of Lemma 10.5. Then we can use SDIT in place of PIT for small degree circuits. The straight-line programs of the elements in S can be assumed to be weakly skew since the determinant has a weakly skew straight-line program [MP]. Q.E.D.

The following is a full statement of Theorem 1.5 (1) and (2).

Theorem 10.7 *Let K be an algebraically closed field of characteristic zero. Let $W = W_n$ be an explicit variety. Then:*

- (a) *An s.s.o.p. exists for $K[W_n]$.*
- (b) *The problem of constructing an s.s.o.p. for $K[W_n]$ belongs to PSPACE. This means an s.s.o.p. can be constructed in $\text{poly}(n)$ work-space, given the circuit C_n specifying W_n as in Definition 10.2.*
- (c) *The problem belongs to $\Sigma^3 \subseteq PH$ assuming GRH.*
- (d) *The problem of constructing an h.s.o.p. for $K[W_n]$ belongs to EXPSPACE.*
- (e) *It belongs to $REXP^{NP}$ if the second fundamental theorem (SFT) holds for $K[W_n]$ as in Definition 10.8 below.*

Analogous statement also holds for an explicit variety in a relaxed sense without any degree restriction.

Definition 10.8 *We say that SFT (Second Fundamental Theorem) holds for an explicit family $\{W_n\}$ of varieties if the ideal of $W_n \subseteq K^m$ has a set Q of generators such that (1) each generator in Q has a straight-line program over \mathbb{Q} and the coordinates z_1, \dots, z_m of K^m of $O(2^{\text{poly}(n)})$ bit-length, and (2) the specification of Q consisting of straight-line programs for its elements can be computed in $O(2^{\text{poly}(n)})$ time and $\text{poly}(n)$ work-space.*

The size of the straight-line programs in (1) is clearly $\Omega(m)$. This size is exponential in n if m is exponential in n , as would be the case in the intended applications. Hence the $\text{poly}(n)$ work-space restriction in (2) is an essentially optimal uniformity condition. For this reason, we say that $\{W_n\}$ has *explicit* defining equations if SFT holds for $\{W_n\}$.

Proof of Theorem 10.7:

(a) Analogue of Theorem 2.1 also holds for strong black-box derandomization. Specifically, the hitting set B in Theorem 2.1 is also a hitting set against all non-zero polynomials that can be approximated infinitesimally closely by arithmetic circuits over K and r variables of size $\leq s$ and degree $\leq d$; cf. Theorem 4.4 in [HS]. Hence the proof of (a) is very similar to that of Theorem 3.9 and 8.18 (c). Specifically, we now use the hitting set B given by Theorem 2.1 (with appropriate parameters) in place of the hitting set T used in the proof of Theorem 10.4 (a). Since the new hitting set B cannot be computed efficiently, what we get now is an s.s.o.p. instead of an e.s.o.p.

(b) and (c): The proof of (a) can now be constructivized very much as in the proof of Theorem 3.11. So we only give a sketch.

Let $F_n(v, x)$ and C_n be as in Definition 10.2. Let $F_n(v, x)_c$ denote the degree c part (in v) of $F_n(v, x)$. We obtain circuits $C_{n,c}$'s for $F_n(v, x)_c$'s from C_n using interpolation as in the proof of Lemma 10.5 (c). Given any potential hitting set T , let S and $h_{b,c}(z)$'s be defined as in eq. (46). Let

$$A = \psi^{-1}(S) = \{\psi^{-1}(h_{b,c}(z)) = F_n(v, b)_c \mid b \in T, 0 < c \leq \deg(F_n)\}.$$

It follows from the proof of Theorem 10.4 (a) that the set S is an s.s.o.p. iff every $f_j(v)$ vanishes on the zero set $Z(A)$ of A . This is so iff every element of $B = \{F_n(v, b')_c \mid b' \in [d+1]^n\}$, $d = \deg(F_n)$, vanishes on $Z(A)$. This last test can be done in $\text{poly}(n)$ work space using the PSPACE-algorithm for Hilbert's Nullstellensatz [Ko, Ko1]. Assuming GRH, it can be done by a Π_2 -algorithm using Theorem 2.14. Now just guess T and carry out this test. By the proof of (a), a correct guess exists.

(d) The proof is similar to that Proposition 8.2.

(e) The proof is similar to that of Proposition 3.1 using the assumed SFT in place of Theorem 2.8. Q.E.D.

This proof also yields:

Theorem 10.9 *The problem of strong black-box derandomization of PIT belongs to PSPACE unconditionally and to Σ_3 assuming GRH.*

10.4 Equivalence

The following is a precise statement of Theorem 1.6.

Theorem 10.10 (Equivalence) *Let K be an algebraically closed field of characteristic zero.*

(a) *The strong black box derandomization of symbolic determinant identity testing (SDIT) over K is equivalent to strict derandomization of Noether's normalization lemma for $\Delta[\det, m]$.*

(b) *The strong black-box derandomization for SPIT is equivalent to strict derandomization of Noether's normalization lemma for $\Delta[\text{perm}, m, m]$ (cf. Example 3 in Section 10.2.1).*

(c) *The strong black-box derandomization of general PIT over K is equivalent to strict derandomization of Noether's normalization lemma for the orbit closure of the P -complete function $H(Y)$ defined in Section 6 of [MS1] (and denoted as $\Delta[H(Y)]$ there).*

(d) The similarly defined strong black-box derandomization of PIT for depth three circuits over K and n variables with degree $\leq d$ and top fan-in $\leq k$ is equivalent to strict derandomization of Noether's Normalization Lemma for the k -th secant variety $X(d, k, n)$ of the Chow variety defined below.

Let S_n^d be the space of degree d homogeneous forms in n variables, and $P(S_n^d)$ the associated projective space. The variety $X(d, k, n) \subseteq P(S_n^d)$ here is, by definition [L1], the projective closure of the set of polynomials that can be expressed as sum of k terms, each term a product of d linear forms.

Proof: We shall only prove (a), the proof of (b), (c), and (d) being similar.

Strong black-box derandomization of SDIT implies strict derandomization of Noether's normalization lemma for $\Delta[\det, m]$ by Theorem 10.4 (b).

Conversely, suppose Noether's normalization lemma for $\Delta[\det, m] \subseteq P(\mathcal{X})$ has strict derandomization. (Here \mathcal{X} , as in Example 3 in Section 10.2.1, is the vector space over K of homogeneous polynomials of degree m in the entries of an $m \times m$ variable matrix Y .) This means the homogeneous coordinate ring $R[\det, m]$ of $\Delta[\det, m]$ has a strict e.s.o.p. S . Let $I[\det, m]$ be the ideal of $\Delta[\det, m]$ so that $R[\det, m] = K[\mathcal{X}]/I[\det, m]$.

Let z_α 's, where α ranges over the monomials in the entries of Y of degree m , be the coordinates of \mathcal{X} . Thus each homogeneous form $h(Y)$ of degree m can be written as $\sum_\alpha z_\alpha(h)\alpha(Y)$, where $z_\alpha(h) \in K$ denote the coordinates of $h \in \mathcal{X}$. If $h(Y)$ is non-zero, we also think of it as a point in $P(\mathcal{X})$.

Since S is strict (cf. Definition 10.3), each element of S is of the form

$$s_b := \sum_\alpha z_\alpha \alpha(b),$$

for some $m \times m$ matrix $b \in Z^{m^2}$ of $\text{poly}(n)$ bit-length, and the specification of S specifies each such b . Let $B = \{b \mid s_b \in S\}$. Its bit-size is clearly $\text{poly}(n)$. It is $\text{poly}(n)$ -time computable since S is.

So we only have to prove that B is a hitting set against symbolic determinants of size m .

Since S is an e.s.o.p., $R[\det, m]$ is integral over the subring generated by S . Hence each z_α satisfies a monic polynomial equation of the form:

$$z_\alpha^k + a_{k-1}z_\alpha^{k-1} + \cdots + a_0 = 0, \text{ mod } I[\det, m],$$

where each a_j is a non-constant homogeneous polynomial in the elements of S . It follows that every element in S cannot vanish at any given (nonzero) $h = h(Y) \in \Delta[\det, m]$. Otherwise, every z_α would vanish at h , and hence, h would be identically zero.

Now suppose a nonzero polynomial $h = h(Y)$ of degree m can be approximated infinitesimally closely by expressions of the form $\det(Y')$, where Y' is an $m \times m$ matrix whose each entry is a homogeneous linear form in the entries of Y with coefficients in K . By the definition of $\Delta[\det, m]$ (cf. Example 3 in Section 10.2.1), it follows that $h(Y)$ considered as a point in $P(\mathcal{X})$ lies in $\Delta[\det, m]$. Since $h(Y)$ is not identically zero, it follows from the above argument that some $s_b \in S$ does not vanish on h ; i.e., $h(b) \neq 0$. This means B is a hitting set against every nonzero

polynomial $h(Y)$ that can be approximately infinitesimally closely by symbolic determinants of size m . In other words, SDIT has strong black-box derandomization. Q.E.D.

11 An approach to derandomization

Theorem 10.10 suggests a natural approach to black-box derandomization of PIT via derandomization of Noether's Normalization Lemma for the associated variety. In this section we formulate a few conjectures which may be helpful in this context.

Let $W \subseteq K^m$ be an explicit variety as in Definition 10.2. Theorem 10.4 leads to:

Conjecture 11.1 *Noether's normalization lemma for the coordinate ring $K[W]$ of any explicit variety W can be derandomized in a strict form (cf. Definition 10.3).*

When $W = \Delta[\det, m]$, this implies black-box derandomization of SDIT (Theorem 10.10). For black-box derandomization of PIT (without degree restrictions), let W be the variety associated with the P -complete function $H(X)$ in [MS1] instead of $\Delta[\det, m]$.

11.1 An explicit Gröbner basis

We now formulate the notion of an explicit Gröbner basis which may be a helpful for proving this conjecture.

Let $y = (y_1, \dots, y_m)$ be any explicit homogeneous system of coordinates for K^m , possibly different from $z = (z_1, \dots, z_m)$ above. By explicit, we mean that each $\psi^*(y_j)$ has a short specification as a straight-line program (as in Definition 10.3 (a)) that can be computed in $\text{poly}(n)$ time. Let $I(W) \subseteq K[y]$ be the ideal of W . Let \prec be the reverse lexicographic term order on the monomials in y_j 's, with $y_1 \prec y_2 \prec \dots \prec y_m$. Let $I_{\prec}(W)$ be the monomial ideal generated by the initial monomials (with respect to \prec) of the polynomials in $I(W)$. The monomials not in $I_{\prec}(W)$ are called *standard*. Let $GB(W) \subseteq I(W)$ be a Gröbner basis [Stm2] of $I(W)$ with respect to the term order \prec . This means the initial monomials of the polynomials in $GB(W)$ generate $I_{\prec}(W)$.

Definition 11.2 *We say that the Gröbner basis $GB(W)$ is explicit if (1) its specification consisting of a straight-line program for every $g(y_1, \dots, y_m) \in GB(W)$ can be computed in $\text{poly}(n)$ work space, (2) every standard monomial involves only $\text{poly}(n)$ y_j 's, (3) given a monomial μ in y_j 's in a sparse representation (only nonzero exponents are specified in binary), whether μ is standard or not can be decided in time that is polynomial in n and the total bit-length of the sparse representation of μ , and (4) for any d , whether there exists a standard monomial of degree d can be decided in $\text{poly}(\langle d \rangle, n)$ time, where $\langle d \rangle$ denotes the bit-length of d .*

By (2) and (3), the Hilbert function $h_W(d)$, the number of standard monomials of degree d , is a $\#P$ -function, and by (4), the problem of deciding if it is non-zero belongs to P . The size of the straight-line programs in (1) is $\Omega(m)$. This is exponential in n if m is exponential in n , as would be the case in the intended applications. Hence the $\text{poly}(n)$ work-space requirement in (1)

is essentially optimal. In contrast, the work-space requirement of the Gröbner basis algorithms [MR2] for general varieties is exponential in n .

An explicit Gröbner basis exists for (the ideal of) the Grassmanian by the straightening algorithm in Chapter 3 of [Stm2]), for Schubert varieties by the straightening algorithm in [ReS], for G/P and their Schubert subvarieties by the straightening algorithm in [GL], and for the explicit toric variety associated with the Birkoff polytope by Theorem 6.1 in [DS] and Theorem 3.1 in [Stm1].

Conjecture 11.3 *Let W be any explicit variety in Theorem 10.10. Then $I(W)$ has an explicit Gröbner basis.*

A plausible approach to black-box derandomization of SDIT is to (1) first construct an explicit Gröbner basis for the ideal $I[\det, m]$ of $\Delta[\det, m]$, and (2) use it derandomize Noether's Normalization Lemma for $\Delta[\det, m]$ (in a strict form), the story for general PIT and PIT for depth three circuits being similar. The step (2) depends on (1), and nothing can be said about it at present since the step (1) itself seems extremely difficult. Hence it would be interesting to carry out these steps as far as possible for easier explicit varieties first, such as V/G in Theorem 1.1, explicit toric varieties, explicit quiver varieties, explicit curves, surfaces, and so on.

11.2 Derandomization of the Grassmanian

With this in mind, we now carry out these steps for one of the simplest explicit varieties—the Grassmanian. The results in this section can also be extended to other determinantal varieties such as G/P and their Schubert sub-varieties. But we focus on the Grassmanian since this illustrates all the basic ideas.

Let G_k^n be the Grassmanian of k -dimensional subspaces in K^n . Any $A \in G_k^n$ can be identified with a $k \times n$ matrix over K whose rows correspond to k basis vectors in A . This matrix depends on the basis, but this choice does not matter in what follows. Hence, by abuse of notation, we denote any such matrix by A again. For any $1 \leq i_1 < i_2 < \dots < i_k \leq n$, let $A[i_1, \dots, i_k]$ denote the $k \times k$ minor of A involving the columns i_1, \dots, i_k . Then $A[i_1, \dots, i_k]$ are the Plücker coordinates of A [Fu]. Consider the Plücker map $\psi : K^{nk} \rightarrow K^t$, $t = \binom{n}{k}$, given by

$$\psi : A \rightarrow (\dots, A[i_1, \dots, i_k], \dots) \subseteq K^t. \quad (47)$$

It is well-known (cf. pages 227 and 384 in [Fu]) that the image of this map is closed, and G_k^n can be identified with the projectivization of this closed image $\text{Im}(\psi)$. Let $z = (\dots, z_{i_1, \dots, i_k}, \dots)$ denote the Plücker coordinates of K^t , so that $\psi^*(z_{i_1, \dots, i_k})(A) = A[i_1, \dots, i_k]$. Let I denote the ideal of $G_k^n = \text{Im}(\psi)$. Then $K[z]/I$ can be identified with the homogeneous coordinate ring $K[G_k^n]$ of the Grassmanian. It is easy to see that the Grassmanian is an explicit variety (cf. Definition 10.2) with the defining Plücker map (47).

Given any $A \in G_k^n$ and any $n - k$ -dimensional subspace L in K^n , fix any $(n - k) \times n$ -matrix representing L (which we denote by L again), and let $[A; L]$ denote the $n \times n$ matrix whose first

k rows are formed by A and the last $(n - k)$ rows are formed by L . By Laplace expansion,

$$\det[A; L] = \sum_{[i_1, \dots, i_k]} A[i_1, \dots, i_k] L[i_1, \dots, i_k],$$

where $L[i_1, \dots, i_k]$ denotes the cofactor of $A[i_1, \dots, i_k]$. This is formed by taking the $(n - k) \times (n - k)$ minor of L formed by the columns with indices different than i_1, \dots, i_k and multiplying it by an appropriate sign, which we denote by $\text{sgn}[i_1, \dots, i_k]$.

Define

$$z_L = \sum_{[i_1, \dots, i_k]} L[i_1, \dots, i_k] z_{i_1, \dots, i_k} \in K[G_k^n], \quad (48)$$

so that for any $A \in G_k^n$,

$$z_L(A) = \det[A; L].$$

Ignoring a constant factor, z_L depends only on L and not on the choice of the matrix representing L .

We say that Noether's normalization lemma for $K[G_k^n]$ is derandomized in a strict form if there exists a $\text{poly}(n)$ -time computable set $S = \{L_1, \dots, L_l\}$ of $(n - k)$ -dimensional subspaces in K^n such that $K[G_k^n]$ is integral over the subring generated by $S' = \{z_{L_1}, \dots, z_{L_l}\}$.

Proposition 11.4 *Noether's Normalization Lemma for $K[G_k^n]$ can be derandomized in a strict form.*

We also note down a consequence of this result in the context of black-box derandomization of the symbolic determinant identity testing (SDIT) of the Grassmanian type.

By a *Grassmanian determinantal projection* of degree n over K we mean the determinant of an $n \times n$ matrix A such that (1) for some $1 \leq k \leq n$, the (i, j) -th entry of A , for $i \leq k, 1 \leq j \leq n$, is a variable $x_{i,j}$, and (2) every other entry of A is an arbitrary nonuniform constant from the base field K . The Grassmanian SDIT is the problem of deciding if $\det(A)$ is identically zero.

The non-black-box version of this problem is easy: $\det(A) \neq 0$ iff the rank of the submatrix of A consisting of rows higher than k is $n - k$. This can be checked fast in parallel over any field [Mul]. We call the SDIT above of the Grassmanian type, because its black-box version is equivalent to (strict) derandomization of Noether's normalization lemma for the Grassmanian G_k^n (cf. Lemma 11.6).

The black-box derandomization problem for SDIT of the Grassmanian type is solved optimally in the following result. As was pointed out to us after this article was written, this has also been proved independently in Forbes and Shpilka [FS] with a different proof technique based on Gabizon and Raz [GR] but essentially the same hitting set.

Proposition 11.5 *Let K be any field that contains at least n^2 distinct elements; e.g. \mathbb{Q} , \mathbb{C} , or a finite field F_p of cardinality at least n^2 . Then there exists an explicit $\text{poly}(n)$ -time computable hitting set against the Grassmanian determinantal projections of degree n over K .*

Equivalently, there exists an explicit $\text{poly}(n)$ -time computable set $S = \{L_1, \dots, L_l\}$, $l = \text{poly}(n)$, of $(n - k)$ -dimensional subspaces in K^n such that for any k -dimensional subspace $A \subseteq$

K^n , there exists an $L_t \in S$ such that $L_t \cap A$ is the origin. All subspaces are assumed to contain the origin.

The cardinality of S here is optimal (equal to $\dim(K[G_k^n])$).

More strongly, S can be computed by a uniform AC^0 circuit of $\text{poly}(n)$ bit-size.

We call the set S in this theorem an *explicit hitting set* against all k -dimensional subspaces of K^n . The optimality is possible here because $K[G_k^n]$ is much simpler than $K[V]^G$ in Theorem 1.1 or 1.3.

We shall show that the following explicit set S is a hitting set. Let $l = k(n - k)$. Let $\lambda_j \in K$, $1 \leq j \leq n$, be distinct elements of $\text{poly}(n)$ bit-lengths, and $\alpha_t \in K$, $1 \leq t \leq l$, distinct nonzero elements with $\text{poly}(n)$ bit-lengths. Let L_t , $1 \leq t \leq l$, be the $(n - k)$ -subspace of K^n spanned by the rows of the $(n - k) \times n$ matrix whose (r, s) -th entry is $\alpha_t^s \lambda_s^r$. Let

$$S = \{L_1, \dots, L_l\}. \quad (49)$$

Clearly S can be computed in $\text{poly}(n)$ time, and also by a uniform AC^0 circuit of $\text{poly}(n)$ bit-size. Hence it is explicit.

If K is not algebraically closed, replace it by its algebraic closure. Certainly S is a hitting set over K if it is a hitting set over its algebraic closure. Hence, without loss of generality, we can assume that K is algebraically closed.

Proposition 11.5 follows from Proposition 11.4 in view of the following equivalence.

Lemma 11.6 (Equivalence) *Black-box derandomization of the Grassmanian SDIT is equivalent to derandomization of Noether's normalization lemma for $K[G_k^n]$ in a strict form. Specifically, a set $S = \{L_1, \dots, L_l\}$ of $(n - k)$ -dimensional subspaces in K^n is a hitting set against all k -dimensional subspaces of K^n iff $K[G_k^n]$ is integral over the subring generated by $S' = \{z_{L_1}, \dots, z_{L_l}\}$.*

Proof: Suppose $S = \{L_1, \dots, L_l\}$ is a hitting set against all k -dimensional subspaces of K^n . This means for every k -dimensional space $A \in G_k^n$ there exists L_j , $j \leq l$, such that $z_{L_j}(A) = \det[A; L_j] \neq 0$. Since the image of the Plücker map ψ (cf. (47)) is closed and coincides with G_k^n , it follows that $G_k^n \cap Z(S') = \{0\}$, where G_k^n is embedded in K^t as in (47), $Z(S') \subseteq K^t$ denotes the zero set of S' , and 0 denotes the origin in K^t . Now it follows from Hilbert's Nullstellensatz and the graded Noether's Normalization Lemma (cf. Lemma 2.10) that $K[G_k^n]$ is integral over the subring generated by S' .

Conversely, suppose $K[G_k^n]$ is integral over the subring generated by $S' = \{z_{L_1}, \dots, z_{L_l}\}$. Then we claim that S is a hitting set against all k -dimensional subspaces $A \in G_k^n$. Suppose to the contrary that there exists an $A \in G_k^n$ such that $\det[A; L_t] = 0$ for all $1 \leq t \leq l$. Then $z_{L_t}(A) = \det[A; L_t] = 0$ for all t . Since each z_{i_1, \dots, i_k} is integral over the subring generated by S' , it satisfies a monic equation of the form

$$z_{i_1, \dots, i_k}^r + a_{r-1} z_{i_1, \dots, i_k}^{r-1} + \dots + a_1 z_{i_1, \dots, i_k} + a_0 = 0,$$

where each a_j is a non-constant homogeneous polynomial in the ring generated by S' . Hence, each a_j vanishes at A . It follows that each z_{i_1, \dots, i_k} vanishes at A . That is, every Plücker

coordinate $A[i_1, \dots, i_k]$ of A is zero, and hence the rank of A is less than k ; a contradiction. Q.E.D.

Let $S = \{L_1, \dots, L_l\}$, $l = k(n-k)$, be as in (49) and let $S' = \{z_{L_1}, \dots, z_{L_l}\}$. Proposition 11.4 follows from the following result.

Lemma 11.7 *The coordinate ring $K[G_k^n]$ of the Grassmanian G_k^n is integral over the subring generated by S' .*

Since $l = \dim(K[G_k^n])$, this means S' is also an h.s.o.p. of $K[G_k^n]$.

The proof of this result is based on the existence of an explicit Gröbner basis for the ideal of the Grassmanian (cf. Chapter 3 in [Stm2]), which implies that its coordinate ring is a Hodge algebra [DEP2]. As such it illustrates an approach to derandomization in Section 11.1 based on explicit Gröbner basis in the simplest setting.

Let P denote the set of Plücker coordinates of G_k^n . Make it a poset by setting $z_{i_1, \dots, i_k} \leq z_{j_1, \dots, j_k}$ iff $i_q \leq j_q$ for all q . The height of P , i.e., the length of a longest chain in P , coincides with $l = \dim(K[G_k^n]) = k(n-k)$. It is known [DEP2] that $K[G_k^n]$ is a Hodge algebra over P (this is a consequence of an explicit Gröbner basis for the ideal of G_k^n [Stm2]). This means:

(1) The standard monomials in the Plücker coordinates form a basis of $K[G_k^n]$. Here we call a monomial in the Plücker coordinates *standard* if it is of the form $z_{i_1^1, \dots, i_k^1} z_{i_1^2, \dots, i_k^2} \cdots$, where $[i_1^1, \dots, i_k^1] \leq [i_1^2, \dots, i_k^2] \leq \cdots$.

(2) If μ is a nonstandard monomial in the Plücker coordinates and

$$\mu = \sum_j a_j \mu_j, \quad a_j \in K,$$

is its unique expression as a linear combination of distinct standard monomials μ_j 's, then for each Plücker coordinate z_{i_1, \dots, i_k} that divides μ and for each μ_j , there exists a Plücker coordinate $z_{i'_1, \dots, i'_k}$ that divides μ_j and satisfies $z_{i'_1, \dots, i'_k} < z_{i_1, \dots, i_k}$.

We will deduce Lemma 11.7 from the following result based on the Hodge algebra structure of $K[G_k^n]$.

Theorem 11.8 (DeConcini, Eisenbud, Procesi) (cf. Theorem 6.3. in [DEP2]) *For any c , $1 \leq c \leq l = k(n-k)$, let*

$$z_c = \sum_{[i_1, \dots, i_k]} \alpha(a) z_{i_1, \dots, i_k},$$

where $[i_1, \dots, i_k]$ ranges over all Plücker indices of height c in the poset P , and each $\alpha(a)$ is any nonzero element of K . Then $\{z_c\}$ forms an h.s.o.p. of $K[G_k^n]$.

Here the height of a Plücker index $[i_1, \dots, i_k]$ is $\sum_{j=1}^k (i_j - j)$. In [DEP2], this result is stated with each $\alpha(a) = 1$. The proof for general nonzero $\alpha(a)$'s is very similar and hence is omitted. When each $\alpha(a) = 1$ as in [DEP2], z_c 's are $P^{\#P}$ -computable. In contrast, the elements of S' in Lemma 11.7 are P -computable and even NC -computable.

Proof of Lemma 11.7:

Let $X \in G_k^n$ be a generic k -dimensional subspace. For any L_t , $t \leq l$, let $Y_t = [X; L_t]$. Identify the Plücker coordinate z_{i_1, \dots, i_k} with $X[i_1, \dots, i_k]$. Thus

$$z_{L_t} = \det(Y_t) = \sum_{[i_1, \dots, i_k]} L[i_1, \dots, i_k] X[i_1, \dots, i_k] \in K[G_k^m], \quad (50)$$

and

$$S' = \{\det(Y_t)\}. \quad (51)$$

Let $\Delta(y_1, \dots, y_r)$ denote the determinant of the $r \times r$ Van-der-Monde matrix whose (i, j) -th entry is y_j^i . For our specific choice of L_t ,

$$L_t[i_1, \dots, i_k] = \alpha_t^{\sum_{s=1}^n j_s} \operatorname{sgn}[i_1, \dots, i_k] \Delta(\lambda_{j_1}, \dots, \lambda_{j_{n-k}}) = \alpha_t^{\sum_{s=1}^n s - \sum_{r=1}^k i_r} \operatorname{sgn}[i_1, \dots, i_k] \Delta(\lambda_{j_1}, \dots, \lambda_{j_{n-k}}), \quad (52)$$

where $1 \leq j_1 < \dots < j_{n-k} \leq n$ are the indices that form the complement of $i_1 < \dots < i_k$.

For $1 \leq c \leq l = k(n-k)$, let

$$z_c = \sum_{[i_1, \dots, i_k]} \operatorname{sgn}[i_1, \dots, i_k] \Delta(\lambda_{j_1}, \dots, \lambda_{j_{n-k}}) X[i_1, \dots, i_k],$$

where $[i_1, \dots, i_k]$ ranges over the Plücker indices of height c , and $[j_1, \dots, j_{n-k}]$ denotes the complement of $[i_1, \dots, i_k]$ in $[1, \dots, n]$.

By (52), for any Plücker index $[i_1, \dots, i_k]$ of height c ,

$$\begin{aligned} L_t[i_1, \dots, i_k] &= \alpha_t^{(\sum_{s=1}^n s) - (\sum_{r=1}^k i_r) - c} \operatorname{sgn}[i_1, \dots, i_k] \Delta(\lambda_{j_1}, \dots, \lambda_{j_{n-k}}) \\ &= \alpha_t^{n(n+1)/2 - k(k+1)/2} \alpha_t^{-c} \operatorname{sgn}[i_1, \dots, i_k] \Delta(\lambda_{j_1}, \dots, \lambda_{j_{n-k}}). \end{aligned}$$

Hence, by eq.(50), for any $1 \leq t \leq l = k(n-k)$,

$$z_{L_t} = \det(Y_t) = \alpha_t^{n(n+1)/2 - k(k+1)/2} \sum_{c=1}^l \frac{1}{\alpha_t^c} z_c.$$

Let \bar{Y} be the vector whose t -th entry is $\det(Y_t)$, and \bar{z} a vector whose c -th entry is z_c . Then

$$\bar{Y} = \alpha_t^{n(n+1)/2 - k(k+1)/2} A \bar{z},$$

where A is the Van-der-Monde matrix whose (t, c) -th entry is $1/\alpha_t^c$.

It follows that

$$\bar{z} = \alpha_t^{k(k+1)/2 - n(n+1)/2} A^{-1} \bar{Y}. \quad (53)$$

Thus z_c 's are linear combinations of $\det(Y_t)$'s. Since the set $\{z_c\}$ is an h.s.o.p by Theorem 11.8, it follows that $S' = \{\det(Y_t)\}$ is also an h.s.o.p. This proves Lemma 11.7. Q.E.D.

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